A polyhedral study of multilinear programs with box constraints

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Abstract

We study the polyhedral convex hull of a mixed-integer set $S$ defined by a collection of multilinear equations of the form $y_I = \prod_{i \in I} x_i$ over the $0$–$1$-cube. Such sets appear frequently in the factorable reformulation of mixed-integer nonlinear optimization problems. In particular, the set $S$ represents the feasible region of a linearized unconstrained zero-one polynomial optimization problem. We define an equivalent hypergraph representation of the mixed-integer set $S$, which enables us to derive several families of facet-defining inequalities, structural properties, and lifting operations for its convex hull in the space of the original variables. Our theoretical developments extend several well-known results from the Boolean quadric polytope and the cut polytope literature, paving a way for devising novel optimization algorithms for nonconvex problems containing multilinear sub-expressions.

1 Introduction

We consider the box-constrained mixed-integer multilinear optimization problem with $n$ variables

\[(ML) \quad \max \left\{ \sum_{I \in \mathcal{I}} c_I \prod_{i \in I} x_i : x_i \in [0,1] \ \forall i \in J_1, \ x_i \in \{0,1\} \ \forall i \in J_2 \right\},\]

where $\mathcal{I}$ is a family of subsets of $N = \{1, \ldots, n\}$, and $c_I, I \in \mathcal{I}$ are nonzero real-valued coefficients. We refer to $r = \max\{|I| : I \in \mathcal{I}\}$ as the degree of Problem (ML). Moreover, the index sets $J_1$ and $J_2$ form a partition of $N$; i.e., $J_1 \cup J_2 = N$ and $J_1 \cap J_2 = \emptyset$.

Problem (ML) subsumes several well-known $\mathcal{NP}$-hard optimization problems. For instance, by letting $J_1 = \emptyset$, Problem (ML) reduces to Pseudo-Boolean optimization (c.f. [12] for an extensive literature review). In addition, since $x_i^p = x_i$, for $p \in \mathbb{Z}_+$ and $x_i \in \{0,1\}$, in this case, Problem (ML) is equivalent to the unconstrained zero-one polynomial optimization. In particular, if $r = 2$, then we obtain the well-studied zero-one quadratic optimization or the max-cut problem (c.f. [6, 21, 7, 8, 19]). More generally, it is well-known that any real-valued function in $0$–$1$ variables can be rewritten as a multilinear function in the same variables. Thus, Problem (ML) subsumes any unconstrained zero-one nonlinear optimization problem [4]. At the other end of the spectrum, by letting $J_2 = \emptyset$ and noting that multilinear functions are closed under scaling and shifting of variables, Problem (ML) is equivalent to maximizing a multilinear

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function over a box. The latter problem has been studied extensively by the global optimization community (c.f. [1, 30, 32, 31, 27, 35, 26, 5, 14]).

A standard technique to tackle Problem (ML) is to first introduce a new variable \(y_I\) for every product term \(\prod_{i \in I} x_i\) with \(|I| \geq 2\) and obtain an equivalent optimization problem in the lifted space \((x, y)\):

\[
\text{(EML)} \quad \max \sum_{I \in \mathcal{I}, |I| = 1} c_I x_I + \sum_{I \in \mathcal{I}, |I| > 1} c_I y_I \\
\text{s.t.} \quad y_I = \prod_{i \in I} x_i, \quad \forall I \in \mathcal{I}, |I| \geq 2 \\
x_i \in [0, 1], \forall i \in J_1 \\
x_i \in \{0, 1\}, \forall i \in J_2.
\]

Subsequently, the feasible region of Problem (EML) is replaced by a convex relaxation and the resulting problem is solved to obtain an upper bound on the optimal value of Problem (ML). A widely-used method to convexify the above problem is to relax the nonconvex region defined by each term \(y_I = \prod_{i \in I} x_i\) over the unit hypercube by its convex hull [20]. Crama [15] derives necessary and sufficient conditions under which the upper bound given by this so called standard linearization is equal to the optimal value of the original problem (EML). However, in general, the standard linearization can lead to very weak bounds [26, 5].

A key observation toward constructing a sharper relaxation for Problem (EML) is that for any vector \(c \in \mathbb{R}^I\), there exists an optimal solution that is attained at a vertex of the unit hypercube. It then follows that the convex hull of the feasible region of (EML) is a polytope and the projection of its vertices onto the space of \(x\) variables is given by \(\{0, 1\}^n\) (c.f. [34]). Consequently, the objective function of Problem (EML) can be equivalently optimized over the following binary set

\[
S = \left\{(x, y) : y_I = \prod_{i \in I} x_i, I \in \mathcal{I}, |I| \geq 2, x \in \{0, 1\}^n\right\}. \tag{1}
\]

The general convexification techniques developed over the past few decades by Sherali and Adams [33], Lovász and Schrijver [25], Balas, Ceria and Cornuéjols [3], Parrilo [29], and Lasserre [23] provide automated mechanisms for the generation of sharp relaxations for integer and nonlinear problems in an extended space. The general idea is to construct hierarchies of successive polyhedral or semidefinite relaxations, whose projection onto the space of original variables converges to the convex hull of the feasible set. For a nonconvex set with a polyhedral convex hull such as the set \(S\) defined above, these techniques result in an exact description of the convex hull after a finite number of steps.

For the quadratic sets, i.e., \(r = 2\), the convex hull of \(S\) is the Boolean quadric polytope \(\text{QP}^n\), defined by Padberg. In [28], Padberg studies various structural properties of \(\text{QP}^n\) and derives several families of facet-defining inequalities as well as lifting operations for this polytope. Moreover, a significant amount of research has been devoted to studying the facial structure of the cut polytope [9, 18, 11], a polytope that is in a one-to-one correspondence with the Boolean quadric polytope [16]. These theoretical developments have had a significant impact on the performance of branch-and-cut based algorithms for mixed-integer quadratic optimization problems [17, 36, 22, 24]. However, for a nonconvex set \(S\) with \(r > 2\), similar polyhedral studies do not exist in the literature. In [13], the authors propose a reduction scheme to reformulate a polynomial zero-one optimization problem to a quadratic one in a higher dimensional space in order to make use of the existing separation algorithms for the Boolean quadric polytope.
and the cut polytope. The proposed approach is most effective when the original instance is reducible; that is, every set in \( I \) containing more than one element is a union of two other sets in \( I \). Otherwise, all such variables are added to the model to make the set \( S \) reducible. In [10], the authors review some quadratization techniques for higher degree multilinear optimization problems and demonstrate their usefulness in some computer vision applications.

Our work is mainly inspired by Padberg’s results on the Boolean quadric polytope. We consider the nonconvex set \( S \) defined by (1) with the degree \( r \) greater than two, and study the facial structure of its polyhedral convex hull in the space of the original variables \((x, y)\). To this end, we define an equivalent hypergraph representation for \( S \). Recall that a hypergraph \( G \) is a pair \((V, E)\) where \( V = V(G) \) is the set of nodes of \( G \), and \( E = E(G) \) is a set of subsets of \( V \) of cardinality at least two, called the edges of \( G \). Throughout this paper, we consider hypergraphs without loops (edges containing a single node) and parallel edges (multiple edges containing the same set of nodes). With any hypergraph \( G \), we associate a set \( S_G \) defined as follows:

\[
S_G = \left\{ z \in \{0, 1\}^d : z_e = \prod_{v \in e} z_{\{v\}}, e \in E(G) \right\},
\]

where \( d = |V(G)| + |E(G)| \). We denote by \( MP_G \) the polyhedral convex hull of \( S_G \). Note that the variables \( z_{\{v\}}, v \in V(G) \) in (2) correspond to the variables \( x_i, i \in N \) in (1) and the variables \( z_e, e \in E(G) \) in (2) correspond to the variables \( y_I, I \in I \) in (1).

Lifting is a widely-used methodology to construct valid inequalities for high-dimensional sets starting from inequalities valid for simpler subsets of the original set. More formally, in our context, consider two sets \( S_G \) and \( S_G' \), as defined by (2), and suppose that \( S_G' \) is obtained from \( S_G \) by letting \( z_{\{v\}} = 0 \) or \( z_{\{v\}} = 1 \) for some \( v \in V(G) \) and/or by projecting out some variables \( z_e, e \in E(G) \). Denote by \( az \leq \alpha \) a valid inequality for \( MP_G \). Lifting \( az \leq \alpha \) means finding a pair \((\bar{a}, \bar{a})\) such that \( \bar{a}z \leq \bar{\alpha} \) is a valid inequality for \( MP_{G'} \), where \( \bar{a} \) is obtained by adding new coordinates to \( a \), after possibly changing some of its coefficients. Given a facet-defining inequality for \( MP_{G'} \), it is often desirable to generate a facet of \( MP_G \) via lifting.

In this paper, we develop the theory of various lifting operations for the polytope \( MP_G \). First, we consider the so called zero-lifting operation, whereby we let \( \bar{a} = (a, 0) \) and \( \bar{\alpha} = \alpha \). As we will show later, in this case, without loss of generality, we can assume that the set \( S_{G'} \) is obtained by letting \( z_{\{v\}} = 0 \) for all \( v \in V(G) \setminus V(G') \). Subsequently, we characterize cases for which the zero-lifting of a facet-defining inequality for \( MP_{G'} \) defines a facet of \( MP_G \). Our results generalize Padberg’s zero-lifting theorems for the Boolean quadratic polytope to sets containing high-degree multilinear. In principle, one could start from multiple inequalities that are facet-defining for distinct low-dimensional sets and lift them simultaneously to obtain a facet of \( MP_G \).

For instance, given a hypergraph \( G \), consider a partition of its nodes defined as \( V(G) = V_1 \cup V_2 \). For \( i = 1, 2 \), let \( G_i \) denote a hypergraph containing edges of the form \( e \cap V_i \) for all \( e \in E(G) \) such that \( e \setminus V_i \neq \emptyset \). Starting from two inequalities that induce facets of \( MP_{G_i} \), for \( i = 1, 2 \), under certain assumptions, we obtain a valid inequality for \( MP_G \) by multiplying these facets and linearizing the resulting relation. Subsequently, we derive conditions under which the new inequality defines a facet of \( MP_G \). We then study a different lifting operation, for which the set \( S_{G'} \) is generated by fixing certain variables in \( S_G \) to one. As we detail later, the hypergraph \( G' \) is obtained by “removing” some nodes from the hypergraph \( G \). Together with the known families of facet-defining inequalities for the Boolean quadratic polytope, the proposed lifting operations enable us to construct many types of facet-defining inequalities for sets containing higher degree multilinear.

The structure of the paper is as follows. In Section 2, we establish a number of fundamental properties of the polytope \( MP_G \), which will be used in the rest of the paper. We develop the theory of zero-lifting for \( MP_G \) in Section 3 and investigate a number of special cases for which the general assumptions are satisfied or can be significantly simplified. In Section 4, we
introduce a facet generation framework, in which certain facets of $MP_G$ can be obtained by multiplying and linearizing facets of simpler polytopes. Lifting via node addition is the subject of Section 5. Finally, conclusions are offered in Section 6.

2 Basic properties of the polytope $MP_G$

In this section, we establish a number of fundamental properties of the polytope $MP_G$ that are essential for the subsequent developments. We start by introducing some graph-theoretic terminology which will be used throughout the paper. Let $G = (V,E)$ be a hypergraph. The rank of $G$ is the maximum cardinality of an edge in $E$. An important special case is when $E$ consists of all subsets of $V$ of cardinality between 2 and $r$, for some $r \geq 2$. We refer to such a hypergraph as a rank-$r$ full hypergraph, and we denote it by $K_{n,r}$, where $n = |V(K_{n,r})|$. Moreover, in this case, we denote the associated nonconvex set (2) and its convex hull by $S_{n,r}$ and $MP_{n,r}$, respectively. In particular, the set $MP_{n,2}$ represents the well-studied Boolean quadric polytope $QP_n$ [28]. A rank-$n$ full hypergraph with $n$ nodes is said to be complete.

A hypergraph $G' = (V',E')$ is a subhypergraph of $G$, if $V' \subseteq V$ and $E' \subseteq E$. Given a subset $V'$ of $V$, the subhypergraph of $G$ induced by $V'$ is the hypergraph $G' = (V',E')$, where $E' = \{ e \in E : e \subseteq V' \}$. In this case, we refer to $G'$ as an induced subhypergraph of $G$. A subset $V' \subseteq V$ is called inducing, if for every $e \in E$ with $|e \cap V'| \geq 2$, we have $e \cap V' \in E$. The support hypergraph of a valid inequality $az \leq \alpha$ for $MP_G$, is the hypergraph $G(a)$, where $V(G(a)) = \{ v \in V : a_{\{v\}} \neq 0 \} \cup \{ v \in V : \exists e \in E \text{ s.t. } v \in e, a_e \neq 0 \}$, and $E(G(a)) = \{ e \in E : a_e \neq 0 \}$. For notational simplicity, we define $L(G) = \{ \{ v \} : v \in V(G) \}$; furthermore, for any vector $z$ having a component $z_{\{v\}}$ corresponding to a node $v$, we write $z_v$, instead of $z_{\{v\}}$. Finally, given $U \subseteq V$, throughout the paper, $w^U$ denotes the $((|V|+|E|))-vector having entries one corresponding to nodes in $U$ and edges $e \in E$ such that $e \subseteq U$, and the remaining entries equal to zero.

We begin by determining the dimension of $MP_G$.  

Proposition 1. The polytope $MP_G$ is full-dimensional for every hypergraph $G$, i.e., $\dim(MPG) = |V(G)| + |E(G)|$.

Proof. The set of $|V(G)| + |E(G)|$ + 1 vectors $w^U$, for every $U \in \{\emptyset\} \cup L(G) \cup E(G)$ are affinely independent vectors in $MP_G$. ⊓⊔

In particular, Proposition 1 implies that the dimension of $\dim(MPG^{n,r}) = \sum_{i=1}^{r} \binom{n}{i}$. Clearly, any inequality of the form $z_p \geq 0$, with $p \in L(G) \cup E(G)$, is valid for $MP_G$. The next proposition provides the condition under which $z_p \geq 0$ is defines a facet of $MP_G$.

Proposition 2. Let $G$ be a hypergraph and let $p \in L(G) \cup E(G)$. Then the inequality $z_p \geq 0$ is facet-defining for $MP_G$ if and only if there exists no $e \in E(G)$ such that $e \supseteq p$.

Proof. Suppose that there exists no edge $e \in E(G)$ such that $e \supseteq p$. Then the vectors $w^U$, for $U \in \{\emptyset\} \cup L(G) \cup E(G) \setminus \{p\}$, are $|V(G)| + |E(G)|$ affinely independent vectors in $S_G$ that satisfy $z_p = 0$. Thus, $z_p \geq 0$ defines a facet of $MP_G$. Conversely, suppose that there exists an edge $e \in E(G)$ such that $e \supseteq p$. Then the two inequalities $z_e \leq z_p$ and $z_e \geq 0$ are valid for $MP_G$ and together imply $z_p \geq 0$, contradicting the assumption that $z_p \geq 0$ is facet-defining. ⊓⊔

The above result implies that $z_v \geq 0$, for some $v \in V(G)$, defines a facet of $MP_G$ if and only if $v$ is an isolated node. If the hypergraph $G$ of Proposition 2 is a rank-$r$ full hypergraph, we have the following characterization:
Proposition 3. The inequality \( z_e \geq 0 \), with \( e \in E(K^{n,r}) \), defines a facet of \( MP^{n,r} \) if and only if \(|e| = r\).

Let \( G' \) be a subhypergraph of a hypergraph \( G \). It is simple to verify that the projection of every vertex of \( MP_G \) onto the space of \( MP_{G'} \) corresponds to a vertex of \( MP_{G'} \), and each vertex of \( MP_{G'} \) can be obtained in this way. The following propositions explain the relationship between the polytopes \( MP_G \) and \( MP_{G'} \) and are consequences of this basic fact.

Proposition 4. Let \( G' \) be a subhypergraph of a hypergraph \( G \). Then \( MP_{G'} \) is obtained from \( MP_G \) by projecting out the variables \( z_v \) with \( v \notin V(G') \), and \( z_e \) with \( e \notin E(G') \).

If \( G' \) is an induced subhypergraph of \( G \), the projection in Proposition 4 can be done in a trivial manner. Given a valid inequality \( az \leq \alpha \) for \( MP_G \), its restriction \( \tilde{a}z \leq \alpha \) to \( MP_{G'} \) is obtained by discarding from \( a \) all components \( a_e \), with \( v \in V(G) \setminus V(G') \) and \( a_e \), with \( e \in E(G) \setminus E(G') \).

Proposition 5. Let \( G' \) be an induced subhypergraph of a hypergraph \( G \), and let \( MP_G = \{ z : a^jz \leq \alpha_j, j \in J \} \). Then \( MP_{G'} = \{ z : \tilde{a}^jz \leq \tilde{\alpha}_j, j \in J \} \).

Proof. The polytope \( MP_{G'} \) can be obtained from the face of \( MP_G \) defined by \( z_v = 0 \), for \( v \in V(G) \setminus V(G') \), and \( z_e = 0 \), for \( e \in E(G) \setminus E(G') \), by projecting out variables \( z_v \), for \( v \in V(G) \setminus V(G') \) and variables \( z_e \), for \( e \in E(G) \setminus E(G') \).

By Proposition 5, if \( G' \) is an induced subhypergraph of a hypergraph \( G \), then the restriction to \( MP_{G'} \) of a valid inequality for \( MP_G \) is valid for \( MP_{G'} \). Note that if \( G' \) contains the support hypergraph of \( az \leq \alpha \) as a subhypergraph, then the restriction \( \tilde{a}z \leq \alpha \) of \( az \leq \alpha \) is obtained by discarding only zero components from \( a \), therefore the two inequalities are identical.

Proposition 6. Let \( az \leq \alpha \) be a valid inequality for \( MP_G \), and let \( G' \) be a subhypergraph of \( G \) containing \( G(a) \) as a subhypergraph. Then the restriction \( \tilde{a}z \leq \alpha \) of \( az \leq \alpha \) to \( MP_{G'} \) is valid for \( MP_{G'} \). Moreover, if \( az \leq \alpha \) is facet-defining for \( MP_G \), then \( \tilde{a}z \leq \alpha \) is facet-defining for \( MP_{G'} \).

Proof. Let \( \tilde{z} \) be a vertex of \( MP_{G'} \), and let \( \tilde{z} \) be a vertex of \( MP_G \) whose projection onto the space of \( MP_{G'} \) is \( \tilde{z} \). Then the validity of \( \tilde{a}z \leq \alpha \) follows from the validity of \( az \leq \alpha \).

Assume now that \( az \leq \alpha \) is facet-defining for \( MP_G \). Then there exists a set \( Z \) of \( |V(G)| + |E(G)| \) affinely independent vertices of \( MP_G \) that satisfy \( az = \alpha \). Let \( \tilde{Z} \) be the projection of \( Z \) onto the space of \( MP_{G'} \). The points in \( \tilde{Z} \) are vertices of \( MP_{G'} \), and they all satisfy \( \tilde{a}z = \alpha \). Moreover they contain \( |V(G')| + |E(G')| \) affinely independent vectors.

Next, we define a switching operation for the set \( MP^{n,r} \) that enables us to convert valid linear inequalities into other valid linear inequalities that induce faces of the same dimension. A similar operator has been introduced by several authors independently in the context of the Boolean quadric polytope and the cut polytope (c.f. [28, 9]).

Consider the hypergraph \( K^{n,r} \), and let \( d = \sum_{i=1}^{r} \binom{n}{i} \). For any \( U \subseteq V(K^{n,r}) \), consider the mapping \( \psi_U : \mathbb{R}^d \to \mathbb{R}^d \) given by

\[
\psi_U(z_v) = \begin{cases} 
1 - z_v & \text{if } v \in U, \\
z_v & \text{if } v \in V(K^{n,r}) \setminus U,
\end{cases}
\]

and

\[
\psi_U(z_e) = \prod_{v \in e \cap U} (1 - z_v) \prod_{v \in e \cap U} z_v, \quad e \in E(K^{n,r}).
\]
By \( z_e = \prod_{v \in e} z_v \), for any \( e \in E(K_n^r) \), relation (4) can be equivalently written as:

\[
\psi_U(z_e) = \sum_{W \subseteq U \atop |W| \text{ even}} z_e \triangleleft U \setminus W - \sum_{W \subseteq U \atop |W| \text{ odd}} z_e \setminus U \setminus W,
\]

where we define \( z_{\emptyset} = 1 \). It is simple to verify that \( \psi_U \) is a non-singular affine transformation as it can be written as \( \psi_U(z) = Az + b \), where \( A \in \mathbb{R}^{d \times d} \) is a lower triangular matrix whose diagonal entries are either 1 or -1. Moreover, for any \( W \subseteq V(K_n^r) \), we have \( \psi_U(w^W) = w^W \Delta U \), where \( W \Delta U = (W \setminus U) \cup (U \setminus W) \) denotes the symmetric difference of \( W \) and \( U \). In particular, \( \psi_U \) maps \( MP_n^r \) onto itself. It follows that the image of a facet-defining inequality for \( MP_n^r \) under \( \psi_U \) is also facet-defining for \( MP_n^r \).

Consequently, by Proposition 3, the following inequalities define facets of \( MP_n^r \):

\[
\psi_U(z_e) \geq 0, \quad \forall e \in E(K_n^r) \text{ with } |e| = n, \forall U \subseteq V(K_n^r).
\]

(5)

The mapping \( \psi_U \) can also be defined for general hypergraphs. For any graph \( G \), and \( U \subseteq V(G) \), \( \psi_U \) is always a non-singular affine transformation, since relation (4) can always be linearized. However, for general hypergraphs this is not always the case, and thus we cannot directly utilize the switching operator to obtain new facets of \( MP_G \) from the existing ones. In [13], the authors characterize the hypergraphs \( G \) for which \( \psi_U \) is a non-singular affine transformation for every \( U \subseteq V(G) \) (see Theorem 4.3 in [13]).

The next theorem follows by a result proven by Sherali and Adams [33].

**Theorem 1** (Theorem 2 in [33]). The polytope \( MP_n^r \) with \( n = r \) is given by facet-defining inequalities (5).

In practice, however, we often have \( n \gg r \) for which \( MP_n^r \) has a far more complex structure. The following result follows directly from Theorem 1 and Proposition 6.

**Corollary 1.** Let \( az \leq \alpha \) define a facet of \( MP_n^r \) and assume that its support hypergraph has \( r \) nodes. Then, \( az \leq \alpha \) is of the form (5).

Note that by Theorem 1, the support hypergraph of any facet-defining inequality for \( MP_n^r \) has at least \( r \) nodes.

We conclude this section by presenting a technical lemma that will be used to prove our main lifting theorems in Sections 3 and 4. Let \( G \) be a hypergraph and let \( G' \) be a subhypergraph of \( G \). Denote by \( az \leq \alpha \) a facet-defining inequality for \( MP_{G'} \) and let \( bz \leq \beta \) denote a valid inequality for \( MP_G \). Suppose that for any point in \( S_G \) whose restriction to \( S_{G'} \) satisfies \( az = \alpha \), we have \( bz = \beta \). The following lemma establishes the relationship between the coefficients of the two inequalities.

**Lemma 1** (Proportionality Lemma). Let \( G = (V, E) \) be a hypergraph and let \( G' = (V', E') \) be a subhypergraph of \( G \) with the following properties:

- \( V' \) is an inducing subset of \( V \),
- any edge of the form \( e \cap V' \) for some \( e \in E \) with \( |e \cap V'| \geq 2 \) and \( e \setminus V' \neq \emptyset \) is present in \( E' \).

Let \( az \leq \alpha \) denote a facet-defining inequality for \( MP_{G'} \) and let \( Q_\alpha = \{ q \in \{0\} \cup L' \cup E' : a_q \neq 0 \} \), where we define \( L' = L(G') \) and \( a_\emptyset = -\alpha \). Let \( bz \leq \beta \) be a valid inequality for \( MP_G \) that is satisfied tightly by any point whose restriction to \( S_{G'} \) satisfies \( az = \alpha \). Then:
1. Let $U$ be a nonempty subset of $V \setminus V'$, and $P_U = \{ p \in L(G) \cup E(G) : p \setminus V' = U \}$. Then, the following cases arise:

(i) if $\{ p \setminus U : p \in P_U \} \supseteq Q_a$, then there exists $\lambda_U \in \mathbb{R}$ such that $b_p = \lambda_U a_p \setminus U$ for all $p \in P_U$ with $p \setminus U \in Q_a$ and $b_p = 0$ for all $p \in P_U$ with $p \setminus U \notin Q_a$.

(ii) otherwise, $b_p = 0$ for all $p \in P_U$.

2. If in addition, we have $b_e = 0$ for all $e \in E \setminus E'$ with $e \subseteq V'$, then $b_p = \lambda a_p$ for all $p \in \{ \emptyset \} \cup L' \cup E'$ for some $\lambda \geq 0$, where we define $b_\emptyset = -\beta$.

Proof. We start by proving part 1. Define $V'' = V \setminus V'$. Let $z^i, i = 1, \ldots, k$, denote the set of all points in $S_{G'}$ satisfying $az = \alpha$. We lift these points to a set of points in $S_G$ by letting $z^i = 0$ for all $v \in V''$ and computing $z^i$, $e \in E \setminus E'$ accordingly. Substituting the lifted points in $bz = \beta$, yields

$$\sum_{p \in V''} b_p z^i_p + \sum_{e \in E'} b_e \prod_{v \in e} z^i_v = \beta, \quad \forall i = 1, \ldots, k,$$

where $E' = \{ e \in E \setminus E' : e \subseteq V' \}$. Next, for every nonempty $U \subseteq V''$, we lift $z^i$, $i = 1, \ldots, k$, to a set of points in $S_{G'}$ by setting $z^i = 1$ for all $v \in U$, $z^i = 0$ for all $v \in V'' \setminus U$ and computing the variables $z^i, e \in E \setminus E'$ accordingly. Define $P_U$ as the (disjoint) union of sets $P_U$, for $W \subseteq U$, $W \neq \emptyset$: i.e., $P_U = \{ p \in L(G) \cup E(G) : p \cap V'' \subseteq U, p \cap V'' \neq \emptyset \}$, where the set $P_U$ is defined in the statement of the theorem. Substituting these points in $bz = \beta$, yields:

$$\sum_{p \in V''} b_p z^i_p + \sum_{e \in E'} b_e \prod_{v \in e} z^i_v + \sum_{p \in P_U} b_p z^i_p = \beta, \quad \forall i = 1, \ldots, k,$$

where we define $\bar{z}^i_\emptyset = 1$. Note that by the inducing assumption on $V'$ and the definition of $E'$, for each $p \in P_U$, we have $p \setminus U \in \{ \emptyset \} \cup L' \cup E'$. From (6) and (7), it follows that

$$\sum_{p \in P_U} b_p z^i_p = 0, \quad \forall i = 1, \ldots, k.$$

We now prove that the following is valid for all nonempty $U \subseteq V''$.

$$\sum_{p \in P_U} b_p z^i_p = 0, \quad \forall i = 1, \ldots, k.$$

We show it by induction on $|U|$, the base case being $|U| = 1$: i.e., $U = \{ u \}$ for some $u \in V''$. In this case, (8) simplifies to (9) since we have $P_U = P_U$. Next, we proceed to the inductive step. Namely, we show that if (9) holds for all $U \subseteq V''$ with cardinality between one and $\delta$, then the same condition is valid for all $U$ with cardinality $\delta + 1$. Consider (8) for a subset $U$ of cardinality $\delta + 1$. Using (9) for all subsets $U$ with cardinality at most $\delta$, the system of equations given by (8) simplifies to (9). Therefore, relation (9) is valid for all nonempty $U \subseteq V''$.

Recall that $z^i, i = 1, \ldots, k$, denote all the points in $S_{G'}$ satisfying the facet-defining inequality $az \leq \alpha$ tightly and in addition these points satisfy $\sum_{p \in P_U} b_p z^i_p = 0$. It follows that for a given $U \subseteq V''$, the equation $\sum_{p \in P_U} b_p z^i_p = 0$ is a scaling of $az - \alpha = 0$. From the definition of $P_U$, it follows that for every $q \in Q_a$, there exists at most one $p \in P_U$ such that $q = p \setminus U$. Note that such a property does not hold for $P_U$, in general. Therefore, the following cases arise:

(i) if $\{ p \setminus U : p \in P_U \} \supseteq Q_a$, then there exists $\lambda_U \in \mathbb{R}$ such that $b_p = \lambda_U a_p \setminus U$ for all $p \in P_U$ with $p \setminus U \in Q_a$ and $b_p = 0$ for all $p \in P_U$ with $p \setminus U \notin Q_a$.

(ii) otherwise, $b_p = 0$ for all $p \in P_U$. 

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We now proceed to part 2 of the lemma. Suppose that $b_e = 0$ for all $e \in E'$. In this case (6) simplifies to
\[
\sum_{p \in L' \cup E'} b_{p} \tilde{z}_{p}^{i} = \beta, \quad \forall i = 1, \ldots, k.
\]
Since $az \leq \alpha$ defines a facet of $MP_{G'}$ and $\tilde{z}^{i}$, $i = 1, \ldots, k$ denotes the set of all points in $S_{G'}$ satisfying this facet tightly, we conclude that $b_{p} = \lambda a_{p}$, for all $p \in L' \cup E'$ and $\beta = \lambda \alpha$ for some $\lambda \in \mathbb{R}$. Let $\tilde{G}$ denote the subhypergraph of $G$ induced by $V'$. By Proposition 5, the restriction of $bz \leq \beta$ to $MP_{\tilde{G}}$ is a valid inequality for $MP_{G'}$. Moreover $bz \leq \beta$ has zero coefficients corresponding to edges not in $E'$. Thus, by Proposition 6, the restriction of $bz \leq \beta$ to $MP_{G'}$ is a valid inequality for $MP_{G'}$. Hence $\lambda \geq 0$.

We are often interested in cases for which the valid inequality $bz \leq \beta$ defines a facet of $MP_{G}$. To this end, we need to make additional assumptions on the structure of the hypergraphs $G$ and $G'$. In the following two sections, we study two important instances for which the inequality $bz \leq \beta$ defines a facet of $MP_{G}$.

### 3 Zero-Lifting

In this section, we develop the zero-lifting theorem for the polytope $MP_{G}$. As we will detail later, our result serves as the generalization of Padberg’s zero-lifting theorem for the Boolean quadric polytope. Let $G'$ be a subhypergraph of a hypergraph $G$. If $az \leq \alpha$ is a valid inequality for $MP_{G'}$, by Proposition 4, we can obtain a valid inequality $\tilde{a}z \leq \alpha$ for $MP_{G}$, by introducing zero coefficients for the additional variables as follows: $\tilde{a}_{p} = a_{p}$ for all $p \in L(G') \cup E(G')$ and $\tilde{a}_{p} = 0$, otherwise. We refer to $\tilde{a}z \leq \alpha$ as the zero-lifting of $az \leq \alpha$ to $MP_{G}$. In the sequel, we say that an inequality $az \leq \alpha$ is nontrivial, if the vector $a$ has at least one nonzero component.

Now suppose that $az \leq \alpha$ defines a facet of $MP_{G'}$. We are interested in characterizing cases for which the zero-lifting of $az \leq \alpha$ defines a facet of $MP_{G}$. Let $\tilde{G}$ be the subhypergraph of $G$ induced by $V(G')$. If the zero-lifting of $az \leq \alpha$ to $MP_{\tilde{G}}$ does not define a facet of $MP_{\tilde{G}}$, then, by Proposition 6, its zero-lifting to $MP_{G}$ is not facet-defining for $MP_{G}$. Thus, in the following, without loss of generality, we assume that the subhypergraph $G'$ is an induced subhypergraph of $G$. We will make use of the following proposition to prove our main lifting result:

**Proposition 7.** Let $G'$ be a subhypergraph of $G$ and let $az \leq \alpha$ be a valid inequality for $MP_{G'}$. Let $U$ be a nonempty subset of $V(G)$ that satisfies the following conditions:

(i) for every nonzero coefficient $a_{p}$, $p \in L(G') \cup E(G')$, we have $p \cup U \in E(G)$,

(ii) if $\alpha$ is nonzero, then $U \in L(G) \cup E(G)$,

(iii) the linearization of the inequality obtained via multiplying $az \leq \alpha$ by $\prod_{v \in U} z_{v}$ is nontrivial and is different from $az \leq \alpha$.

Then, the zero-lifting of $az \leq \alpha$ is not facet-defining for $MP_{G}$.

**Proof.** By multiplying $az \leq \alpha$ by $\prod_{v \in U} z_{v}$ and by $1 - \prod_{v \in U} z_{v}$, and using (i),(ii) to linearize the resulting relations, we obtain two distinct valid linear inequalities for $MP_{G}$, whose sum is $az \leq \alpha$. This implies that $az \leq \alpha$ is not facet-defining for $MP_{G}$. \qed

In the sequel, we say that a valid inequality $az \leq \alpha$ for $MP_{G'}$ is maximal for $MP_{G}$, if there exists no $U \subseteq V(G)$ for which conditions (i)-(iii) of Proposition 7 are satisfied. Now suppose that $G'$ is an induced subhypergraph of $G$ and $V(G')$ is an inducing subset of $V(G)$. In the following lemma, we use these additional assumptions to derive a simpler criterion to check the maximality of a facet of $MP_{G'}$ for $MP_{G}$.

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Lemma 2. Let \( G \) be a hypergraph and let \( G' \) be an induced subhypergraph of \( G \) such that \( V(G') \) is an inducing subset of \( V(G) \). Let the inequality \( az \leq \alpha \) define a facet of \( MP_G \). Then \( az \leq \alpha \) is not maximal for \( MP_G \) if and only if conditions (i),(ii) of Proposition 7 are satisfied for some nonempty \( U \subseteq V(G) \setminus V(G') \).

Proof. First assume that conditions (i),(ii) of Proposition 7 are satisfied for some nonempty \( U \subseteq V(G) \setminus V(G') \). Then the linearized inequalities obtained via multiplying \( az \leq \alpha \) by \( \prod_{v \in U} z_v \) and \( 1 - \prod_{v \in U} z_v \) are nontrivial and are different from the original inequality; i.e., condition (iii) is automatically satisfied. Hence \( az \leq \alpha \) is not maximal for \( MP_G \).

Now assume that \( az \leq \alpha \) is not maximal for \( MP_G \), and let \( U \subseteq V(G) \) satisfy conditions (i)-(iii). Note that in particular \( U \) is nonempty. We will show that the set \( W = U \setminus V(G') \) satisfies (i),(ii). First we show that the inequality obtained via multiplying \( az \leq \alpha \) by \( \prod_{v \in U \cap V(G')} z_v \) is identical to \( az \leq \alpha \). By assumption \( G' \) is an induced subhypergraph of \( G \) and the subset \( V(G') \) is inducing; it follows that for every nonzero \( a_v, p \in L(G') \cup E(G') \), we have \( p \cup U \cap V(G') \subseteq L(G') \cup E(G') \). Moreover, if \( \alpha = 0 \), then \( \alpha \leq \alpha \). Therefore, the inequality \( az \leq \alpha \) obtained by multiplying \( az \leq \alpha \) by \( \prod_{v \in U \cap V(G')} z_v \), can be linearized and is valid for \( MP_G' \). Since \( az \leq \alpha \) defines a facet of \( MP_G' \) and \( W \) is satisfied tightly by every point \( z \in S_G' \) that satisfies \( az = \alpha \), it follows that \( az \leq \alpha \) can be obtained by multiplying \( az \leq \alpha \) by a scalar \( \lambda \geq 0 \).

We are now in a position to present conditions under which the zero-lifting of a facet-defining inequality for \( MP_G' \) is facet-defining for \( MP_G \).

Theorem 2 (Zero-lifting Theorem). Let \( G \) be a hypergraph, let \( G' \) be an induced subhypergraph of \( G \) such that \( V(G') \) is inducing, and suppose that \( az \leq \alpha \) is a facet-defining inequality for \( MP_G' \). Then the zero-lifting of \( az \leq \alpha \) defines a facet of \( MP_G \) if and only if it is maximal for \( MP_G \).

Proof. The necessity of the maximality assumption follows from Proposition 7. Thus, we now show sufficiency. Assume that the zero-lifting \( \bar{a}z \leq \bar{a} \alpha \) of \( az \leq \alpha \) is maximal for \( MP_G \). Denote by \( bz \leq \beta \) a nontrivial valid inequality for \( S_G \) that is satisfied tightly by all points in \( S_G \) satisfying \( \bar{a}z = \bar{a} \alpha \). We show that the two inequalities \( az \leq \alpha \) and \( bz \leq \beta \) coincide up to a positive scaling, implying that \( az \leq \alpha \) defines a facet of \( MP_G \).

All the assumptions of Lemma 1 are satisfied, including the one in Part 2, since \( G' \) is an induced hypergraph of \( G \) and so \( \{ e \in E(G) \setminus E(G') : e \subseteq V(G') \} = \emptyset \). Hence, we have \( b_p = \lambda a_p \) for all \( p \in L(G') \cup E(G') \) and \( \beta = \lambda \alpha \) for some \( \lambda \geq 0 \).

Let \( U \) denote a nonempty subset of \( V(G) \setminus V(G') \). Define \( P_U = \{ p \in L(G) \cup E(G) : p \setminus V(G') = U \} \) and \( Q_a = \{ p \in \{ \emptyset \} \cup L(G') \cup E(G') : a_p \neq 0 \} \), where \( a_p = -a \). Since the inequality \( az \leq \alpha \) is maximal for \( MP_G \), Lemma 2 implies that \( \{ p \setminus U : p \in P_U \} \not\supseteq Q_a \). Consequently, by Part 1 of Lemma 1, we have \( b_p = 0 \) for all \( p \in P_U \) as the above argument applies to every nonempty subset \( U \) of \( V(G) \setminus V(G') \), and \( b_v = 0 \) for all \( v \in V(G) \setminus V(G') \). Hence \( (b, \beta) = (\lambda a, \alpha) \) for some \( \lambda \geq 0 \). By assumption \( bz \leq \beta \) is nontrivial, thus \( \lambda > 0 \), and the theorem follows.

In the reminder of this section, we consider the sets \( S_G \) with certain structures for which the assumptions of the Zero-lifting Theorem are either trivially satisfied or can be simplified significantly.
First, suppose that the hypergraph \( G \) in the statement of Theorem 2 is a rank-\( r \) hypergraph. It follows that, if the facet-defining inequality \( az \leq \alpha \) for \( MP_{G'} \) has a nonzero coefficient corresponding to an edge of \( G' \) of cardinality \( r \), then it is maximal for \( MP_G \). The following lemma implies that, if the induced subhypergraph \( G' \) is rank-\( r \) full, then each facet of \( MP_{G'} \) contains at least one nonzero coefficient corresponding to an edge of cardinality \( r \).

**Lemma 3.** If \( az \leq \alpha \) defines a facet of \( MP^{n',r} \), then \( a_e \neq 0 \) for at least one \( e \in E(K^{n',r}) \) with \( |e| = r \).

**Proof.** By contradiction assume that \( a_e = 0 \) for all \( e \in E(K^{n',r}) \) with \( |e| = r \), and let \( G' = G(a) \). By Proposition 6, the restriction \( \tilde{a}z \leq \alpha \) of \( az \leq \alpha \) to \( MP_{G'} \) is valid for \( MP_{G'} \). Let \( f \) be an edge of maximum cardinality in \( E(G') \), and let \( v \in V(K^{n',r}) \setminus f \). Conditions (i),(ii) of Proposition 7 are satisfied for \( U = \{v\} \), thus \( az \leq \alpha \) is not a facet of \( MP^{n',r} \) by Lemma 2. A contradiction.

By Lemma 3 and Theorem 2, the following result is immediate.

**Corollary 2.** Let \( G \) be a rank-\( r \) hypergraph that contains \( K^{n',r} \) as an induced subhypergraph. Then the zero-lifting of every facet-defining inequality for \( MP^{n',r} \) is facet-defining for \( MP_G \).

**Proof.** Since \( G \) is a rank-\( r \) hypergraph, \( V(K^{n',r}) \) is an inducing subset of \( V(G) \). In addition by Lemma 3, every facet of \( MP^{n',r} \) has a nonzero coefficient corresponding to an edge of \( K^{n',r} \) of rank \( r \), implying its maximality for \( MP_G \). Thus, by Theorem 2 the result follows.

In particular, for rank-\( r \) full hypergraphs, we have the following:

**Corollary 3.** The zero-lifting of every facet of \( MP^{n',r} \) defines a facet of \( MP^{n',r'} \) for all \( n' > n \).

Interestingly, for quadratic sets, i.e., \( MP^{n,r} \) with \( r = 2 \), the results of Theorem 2 and Corollary 3 simplify to the lifting theorems of Padberg (see Corollary 2 and Theorem 3 in [28]). Clearly, the inducing and maximality assumptions of Theorem 2 are trivially satisfied for quadratic sets, but add further restrictions on the lifting operation when the set \( S_G \) contains higher degree multilinear terms.

More generally, for a hypergraph containing a complete subhypergraph, we can state the following lifting result:

**Corollary 4.** Let the hypergraph \( G \) contain a complete subhypergraph \( G' \). The zero-liftings of all facet-defining inequalities for \( MP_{G'} \) are facet-defining for \( MP_G \) if and only if there exists no edge \( e \in E(G) \) such that \( e \supset V(G') \).

**Proof.** Since the subhypergraph \( G' \) is a complete hypergraph, \( V(G') \) is an inducing subset of \( V(G) \). By Lemma 3, for every facet \( az \leq \alpha \) of \( MP_{G'} \), the coefficient \( a_f \), where \( f = V(G') \), is nonzero. Hence, if \( G \) does not have an edge \( e \) of the form \( e \supset V(G') \), we conclude that all facets of \( MP_{G'} \) are maximal for \( MP_G \) and consequently are facet-defining for it by Theorem 2.

Now suppose that \( G \) contains an edge \( e \) such that \( e \supset V(G') \). Let \( \tilde{V} = e \setminus V(G') \). By Proposition 2, the inequality \( z_f \geq 0 \) with \( f = V(G') \) defines a facet of \( MP_{G'} \). After multiplying both sides of this inequality by the nonnegative factor \( \prod_{v \in \tilde{V}} z_v \) and \( 1 - \prod_{v \in \tilde{V}} z_v \) and linearizing the resulting relations we obtain \( z_e \geq 0 \) and \( z_f - z_e \geq 0 \) both of which are valid inequalities for \( MP_G \) and their sum is given by \( z_f \geq 0 \). Thus, \( z_f \geq 0 \) does not define a facet of \( MP_G \). This completes the proof.

Suppose that the induced subhypergraph \( G' \) defined in the statement of Theorem 2 is a non-isolated node; i.e., \( V(G') = \{\tilde{v}\} \) for some \( \tilde{v} \in V(G) \). Clearly, in this case, \( V(G') \) is an inducing subset of \( V(G) \). The convex hull of \( S_{G'} \) is the line segment defined by the two facets \( z_e \geq 0 \) and \( z_{\tilde{v}} \leq 1 \). To characterize the cases for which the zero-lifting of these two inequalities
Proof. By Theorem 2, \( \ell \) simplicity, in the following examples, given a node equivalent to the existence of \( f, g \) if and only if there exists a nonempty \( U \subseteq V(G) \) satisfying conditions (i) and (ii) of Proposition 7. The existence of such a set is equivalent to the existence of an edge of cardinality two containing \( \bar{v} \), if \( |U| = 1 \); and is equivalent to the existence of \( f, g \in E(G) \) with \( f \supseteq g \), and \( f \setminus g = \{ \bar{v} \} \), if \( |U| \geq 2 \).

The result of Corollary 5 implies that \( z_0 \leq 1 \) is not facet-defining for the Boolean quadric polytope, whereas, it defines a facet of a set MP\(_G\), where \( G \) is a \( k \)-uniform hypergraph with \( k \geq 3 \). Recall that a \( k \)-uniform hypergraph is a hypergraph such that all its edges have cardinality \( k \).

Before proceeding further, we demonstrate that the inducing assumption on the subhypergraph \( G' \) of \( G \) is required for the validity of Theorem 2 via a simple example. More precisely, if \( V(G') \) is not an inducing subset of \( V(G) \), it is possible that a facet-defining inequality for MP\(_G\) is maximal for MP\(_G\), and its zero-lifting does not define a facet of MP\(_G\). For notational simplicity, in the following examples, given a node \( v_i \), we write \( z_i \) instead of \( z_{v_i} \). Similarly, given an edge \( \{v_i, v_j, v_k\} \), we write \( z_{ijk} \) instead of \( z_{\{v_i, v_j, v_k\}} \).

Example 1. Consider the set \( S_G \) with \( G = (V, E) \), \( V = \{v_1, v_2, v_3, v_4\} \) and, \( E = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\} \) (see Figure 1). Let \( V' = \{v_1, v_2, v_3\} \) and denote by \( G' \) the subhypergraph of \( G \) induced by \( V' \). The subset \( V' \) is not inducing since the edges \( \{v_1, v_2\} \) and \( \{v_1, v_3\} \) are not present in \( E \). The inequality \( z_{123} \leq z_1 \) is a facet of MP\(_{G'}\) and it contains a nonzero coefficient corresponding to an edge of maximum degree and is therefore maximal for MP\(_G\). We now show that \( z_{123} \leq z_1 \) is not a facet of MP\(_G\) by providing two valid inequalities for MP\(_G\) that together imply \( z_{123} \leq z_1 \). Consider the expression \( \ell = z_{iij}z_j - z_{ijk}z_k + z_{ijk}z_k \) for distinct \( i, j, k \in \{1, \ldots, 4\} \), where \( z \in \{0, 1\} \). It is simple to check that \( \ell \leq 1 \). Now consider \( \ell_1 = z_{223} - z_{224} + z_{224} \) and \( \ell_2 = z_{223} - z_{224} + z_{224} \). Multiplying \( \ell_1 \leq 1 \) and \( \ell_2 \leq 1 \) by \( z_1 \) and linearizing the resulting inequalities we obtain \( z_{123} - z_{124} + z_{134} \leq z_1 \) and \( z_{123} - z_{134} + z_{124} \leq z_1 \). The sum of these two inequalities is \( z_{123} \leq z_1 \), showing that such inequality is not facet-defining for MP\(_G\).

Thus, in general, the inducing assumption on the subhypergraph \( G' \) is required for the validity of Theorem 2. However, for various structured hypergraphs or specific classes of facets, the result of Theorem 2 remains valid even when the inducing assumption is not satisfied. The
this case, $V \setminus V_0$ for MP coefficient corresponding to the edge $\{\}$ subset of $G$ and its zero-lifting define facets of MP.belong to $E_G$ facet-defining for MP

Example 3. Consider the hypergraph $G$, consider the hypergraph $G$, where $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$ (see Figure 2). The set $V' = \{v_1, v_2, v_3\}$ is not inducing since $\{v_2, v_3\}$ does not belong to $E(G)$. Now, let $G'$ be the hypergraph obtained from $G$ by adding the edge $\{v_2, v_3\}$. In this case, $V'$ is an inducing subset of $V(G)$. Let $G''$ denote the subhypergraph of $G'$ induced by $V'$. The inequality $z_{123} \leq z_1$ defines a facet of $MP_{G''}$ and by Theorem 2 is facet-defining for $MP_{G'}$. In $G''$, the zero-lifting of $z_{123} \leq z_1$ is a facet of $MP_{G''}$ and by Theorem 2 is facet-defining for $MP_{G'}$. By Proposition 6, it follows that $z_{123} \leq z_1$ defines a facet of $MP_{G}$ as well, since its coefficient corresponding to the edge $\{v_2, v_3\}$ is zero. 

More generally, we have the following result:

Corollary 6. Let $G$ be a hypergraph and let $G'$ be an induced subhypergraph of $G$. Denote by $az \leq \alpha$ a facet of $MP_G$ that is maximal for $MP_G$. Denote by $G''$ the hypergraph obtained from $G'$ by adding all edges of the form $e \cap V(G')$, where $e \in E(G)$. If the zero-lifting of $az \leq \alpha$ to $MP_{G''}$ is facet-defining for $MP_{G''}$, then its zero-lifting to $MP_G$ defines a facet of $MP_G$.

Proof. Follows directly from Proposition 6 and Theorem 2.

Finally, the following example demonstrates that the converse of Corollary 6 does not hold in general; that is, if $az \leq \alpha$ does not define a facet of $MP_{G''}$, its zero-lifting may still be facet-defining for $MP_G$.

Example 3. Consider the hypergraph $G$, where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_4, v_2, v_5\}\}$ (see Figure 3). Denote by $G'$ the subhypergraph of $G$ induced by the subset $V' = \{v_1, v_2, v_3, v_4\}$. It can be verified that the inequality

$$z_3 - z_{123} + z_{124} \leq 1$$

(10)

and its zero-lifting define facets of $MP_{G'}$ and $MP_G$, respectively. Clearly, $V'$ is not an inducing subset of $V(G)$ since $\{v_1, v_2\}$ does not belong to $E(G)$. Let $G'$ be the hypergraph obtained from
G by adding the edge \{v_1, v_2\} and let \(G''\) denote the subhypergraph of \(\tilde{G}\) induced by \(V'\). We now show that inequality (10) is not facet-defining for \(\text{MP}_{G''}\) by providing two valid inequalities for \(\text{MP}_{G''}\) that together imply (10). Denote by \(H_1\) and \(H_2\) the subhypergraphs of \(G''\) induced by the subsets \(V'_1 = \{v_1, v_2, v_3\}\) and \(V'_2 = \{v_1, v_2, v_4\}\), respectively. It is simple to verify that the inequalities \(z_3 + z_{12} - z_{123} \leq 1\) and \(-z_{12} + z_{124} \leq 0\) define facets of \(\text{MP}_{H_1}\) and \(\text{MP}_{H_2}\), respectively. Hence, their zero liftings are valid inequalities for \(\text{MP}_{G''}\). In addition, adding the two inequalities yields (10). Therefore, the assumption of Corollary 6 is not always required for the validity of the zero-lifting operation. 

4 Lifting via facet multiplication

Let \(G_1\) and \(G_2\) denote two subhypergraphs of a hypergraph \(G\) with \(V(G_1) \cap V(G_2) = \emptyset\) and let the inequalities \(az + \alpha \geq 0\) and \(bz + \beta \geq 0\) define facets of \(\text{MP}_{G_1}\) and \(\text{MP}_{G_2}\), respectively. Suppose that the two inequalities are not maximal for \(\text{MP}_{G}\), implying that their zero-lifting is not facet-defining for \(\text{MP}_{G}\). In particular, assume that for every nonzero \(a_p, p \in L(G_1) \cup E(G_1)\) and \(b_q, q \in L(G_2) \cup E(G_2)\), we have \(p \cup q \in L(G) \cup E(G)\). Clearly, the linearization of the relation \((az + \alpha)(bz + \beta) \geq 0\) is a valid inequality for \(\text{MP}\). We are interested in characterizing the cases for which this inequality defines a facet of \(\text{MP}\). In the special case where \(G, G_1, G_2\) are all complete hypergraphs and \(V(G) = V(G_1) \cup V(G_2)\), by Theorem 1 and relation (4), any linear inequality obtained by multiplying and linearizing two facets of \(\text{MP}_{G_1}\) and \(\text{MP}_{G_2}\) defines a facet of \(\text{MP}_{G}\). Moreover, the collection of all such inequalities characterizes \(\text{MP}_{G}\). In this section, we consider this lifting operation for general sparse hypergraphs. In fact, as we will demonstrate in the following theorem, such lifting operation is valid in a more general setting, namely \(G_1\) and \(G_2\) are auxiliary hypergraphs which are not necessarily subhypergraphs of \(G\).

**Theorem 3.** Let \(G\) be a hypergraph and consider a partition of the nodes of \(G\) defined as \(V(G) = V_1 \cup V_2\). Let

\[
E_i = \{e \cap V_i : e \in E(G), |e \cap V_i| \geq 2, e \setminus V_i \neq \emptyset\}, \quad i = 1, 2.
\]

Define the hypergraphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\). Let the inequalities \(az + \alpha \geq 0\) and \(bz + \beta \geq 0\) define facets of \(\text{MP}_{G_1}\) and \(\text{MP}_{G_2}\), respectively. Finally, suppose that for every nonzero \(a_p, p \in \{\emptyset\} \cup L(G_1) \cup E_1\) and every nonzero \(b_q, q \in \{\emptyset\} \cup L(G_2) \cup E_2\), we have \(p \cup q \in E(G)\), where \(a_\emptyset = \alpha\) and \(b_\emptyset = \beta\). Then the linearization of the relation

\[(az + \alpha)(bz + \beta) \geq 0,
\]

given by

\[
\sum_{p \in L(G_1) \cup E_1} \sum_{q \in L(G_2) \cup E_2} a_p b_q (e_p \cup e_q) + \beta az + \alpha bz + \alpha \beta \geq 0,
\]

defines a facet of \(\text{MP}_{G}\).

**Proof.** Let \(L_1 = L(G_1)\) and \(L_2 = L(G_2)\). We start by defining a hypergraph \(\tilde{G}\) obtained by adding to the hypergraph \(G\) all edges \(e \in E_i, i = 1, 2\) as defined by (11) that are not present in \(E(G)\); i.e., \(V(\tilde{G}) = V(G)\) and \(E(\tilde{G}) = E(G) \cup E_1 \cup E_2\). The key to this construction is that \(V_1\) and \(V_2\) are inducing subsets of \(V(\tilde{G})\), whereas they are not inducing subsets of \(V(G)\), in general. Subsequently, we prove that the zero-lifting of inequality (12) defines a facet of \(\text{MP}_{G}\). It then follows from Proposition 6 that inequality (12) is facet-defining for \(\text{MP}_{G}\) as well, since by assumption its support hypergraph is a subhypergraph of \(G\).

Clearly, inequality (12) is valid for \(\text{MP}_{\tilde{G}}\) as \(G_1\) and \(G_2\) are subhypergraphs of \(\tilde{G}\). Denote by \(cz + \gamma \geq 0\) a nontrivial valid inequality for \(\text{MP}_{\tilde{G}}\) that is satisfied tightly by the set of all
points in $S_G$ that satisfy inequality (12) tightly. We show that the two inequalities coincide up to a positive scaling, which in turn implies inequality (12) defines a facet of $MP_G$. By construction, any point in $S_G$ whose restriction to $S_{G_i}$ (resp. $S_{G_j}$) satisfies $az + \alpha = 0$ (resp. $bz + \beta = 0$), satisfies tightly inequality (12). To characterize the relationship between the coefficients of $az + \alpha \geq 0$ and $cz + \gamma \geq 0$, we first employ the result of Lemma 1 by letting $G' = G_1$. Define $Q_a = \{ p \in \{ 0 \} \cup L_1 \cup E_1 : a_p \neq 0 \}$. Let $U$ denote a nonempty subset of $V_2$ and let $P_U = \{ w \in L(G) \cup E(G) : w \cap V_1 = U \}$. Since $V_1$ is an inducing subset of $V(G)$, by Part 1 of Lemma 1, for each $U \subseteq V_2$, we have:

\begin{align}
(1.1) & \text{ if } p \cup U \in P_U \text{ for all } p \in Q_a, \text{ then there exists } \lambda_U \in \mathbb{R} \text{ such that } c_{p \cup U} = a_p \lambda_U \text{ for all } p \in Q_a, \text{ and } c_{p \cup U} = 0 \text{ for all } p \cup U \in P_U \text{ with } p \notin Q_a, \\
(1.2) & \text{ otherwise, } c_{p \cup U} = 0 \text{ for all } p \subseteq V_1 \text{ such that } p \cup U \in P_U.
\end{align}

Symmetrically, by letting $G' = G_2$ in the statement of Lemma 1, defining $U$ to be a nonempty subset of $V_1$, $P_U = \{ w \in L(G) \cup E(G) : w \cap V_1 = U \}$ and $Q_b = \{ q \in \{ 0 \} \cup L_2 \cup E_2 : b_q \neq 0 \}$, and employing Part 1 of Lemma 1, we obtain:

\begin{align}
(2.1) & \text{ if } U \cup q \in P_U \text{ for all } q \in Q_b, \text{ then there exists } \mu_U \in \mathbb{R} \text{ such that } c_{U \cup q} = \mu_U b_q \text{ for all } q \in Q_b, \text{ and } c_{U \cup q} = 0 \text{ for all } U \cup q \in P_U \text{ with } q \notin Q_b, \\
(2.2) & \text{ otherwise, } c_{U \cup q} = 0 \text{ for all } q \subseteq V_2 \text{ such that } U \cup q \in P_U.
\end{align}

To characterize the coefficients of $cz + \gamma \geq 0$, we partition $L(G) \cup E(G)$ into the following subsets and analyze each separately: (i) $E_{1,2}$ containing any edge $e \in E(G)$ whose intersection with both sets $V_1$ and $V_2$ is nonempty, (ii) $E_1 = \{ e \in E(G) : V_1 \supseteq e, e \notin E_1 \}$ and $E_2 = \{ e \in E(G) : V_2 \supseteq e, e \notin E_2 \}$, (iii) $L_1 \cup E_1$ and $L_2 \cup E_2$.

Consider an edge $e \in E_{1,2}$; define $p = e \cap V_1$ and $q = e \cap V_2$. By our assumption on the structure of $G$, it follows that $p \in L_1 \cup E_1$ and $q \in L_2 \cup E_2$. Let $\mu_p \in \mathbb{R}$ and $\lambda_q \in \mathbb{R}$. We show that

\[ c_e = \begin{cases} 
  \mu_p \lambda_q, & \text{if } p \in Q_a \setminus \{ 0 \}, q \in Q_b \setminus \{ 0 \} \\
  0, & \text{otherwise.}
\end{cases} \tag{13} \]

First, let $p \in Q_a \setminus \{ 0 \}$ and $q \in Q_b \setminus \{ 0 \}$. By assumption, for every nonzero $a_p$, $\tilde{p} \in L_1 \cup E_1$ and every nonzero $b_q$, $\tilde{q} \in L_2 \cup E_2$, we have $p \cup q \in E(G)$ and $c_e = a_p \lambda_q$. Similarly, for any $p \in Q_a \setminus \{ 0 \}$, we have $p \cup q \in E(G)$ for all $p \in Q_a$, and by (1.1), we obtain $c_e = a_p \lambda_q$. Finally, if $p \notin Q_a$ (or $q \notin Q_b$), then by (1.1-2) (resp. (2.1-2)), we have $c_e = 0$. Combining these arguments, we obtain (13).

Next we characterize the coefficients $c_e$ of $cz + \gamma \geq 0$ for all $e \in E_1 \cup E_2$, where the subsets $E_1$ and $E_2$ are as defined above. Since $P_e = \{ e \}$ for all $e \in E_1 \cup E_2$, by (1.2) and (2.2) above we have

\[ c_e = 0, \quad \forall e \in E_1 \cup E_2. \tag{14} \]

Finally, we characterize the remaining coefficients of $cz + \gamma \geq 0$; i.e., $c_w$ for all $w \in L_1 \cup L_2 \cup E_1$. To this end, we utilize the result of Part 2 of Lemma 1 by first letting $G' = G$ and using the fact that $c_e = 0$ for all $e \in E_1$. It then follows that

\[ c_p = \eta a_p, \quad \forall p \in \{ 0 \} \cup L_1 \cup E_1, \tag{15} \]

for some $\eta \geq 0$, where for notational simplicity we define $c_0 = \gamma$. Symmetrically,

\[ c_q = \zeta b_q, \quad \forall q \in \{ 0 \} \cup L_2 \cup E_2, \tag{16} \]
Figure 4: Hypergraphs $G, G_1, G_2$ of Example 4 to demonstrate the lifting scheme introduced in Theorem 3. Namely, a facet of $\text{MP}_G$ can be obtained by multiplying and linearizing certain facets of $\text{MP}_{G_1}$ and $\text{MP}_{G_2}$ for some $\zeta \geq 0$.

To summarize, let us define $Q_{a,b} = \{p \cup q : p \in Q_a, q \in Q_b\}$; i.e., $Q_{a,b}$ consists of those elements of $\{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$ whose corresponding coefficients in inequality (12) are nonzero. Then, for any $w \in \{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$, by relations (13) – (16), we have

$$c_w = \begin{cases} a_p \lambda_q = \mu_p b_q, & \text{if } w = p \cup q, \text{ such that } p \cup q \in Q_{a,b} \\ 0, & \text{otherwise,} \end{cases}$$

(17)

where we define $\mu_p = \zeta$, if $p = \emptyset$ and $\lambda_q = \eta$, if $q = \emptyset$.

Denote by $Q_c$ the set containing those elements of $\{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$ whose corresponding coefficients in $cz + \gamma \geq 0$ are nonzero. By (17), $Q_c \subseteq Q_{a,b}$. We now show that $Q_c = Q_{a,b}$. To do so, it suffices to prove that for any nonzero $a_p$ and nonzero $b_q$ as defined in (17), the coefficient $c_{p,q}$ is nonzero as well. Assume the contrary by letting $c_{p,q} = 0$ for some $p \cup q \in Q_{a,b}$. Since $a_p \neq 0$ and $b_q \neq 0$, by (17), we have $\mu_p = \lambda_q = 0$. It follows that $c_{p,q} = \mu_p b_q = 0$ for all $q \in Q_b$. By (17), $c_{p,q}$ can be equivalently written as $c_{p,q} = a_p \lambda_q$ and since by assumption $a_p \neq 0$, it follows that $\lambda_q = 0$ for all $q \in Q_b$. Consequently, $c_{p,q} = a_p \lambda_q = 0$ for all $p \cup q \in Q_{a,b}$; i.e., $cz + \gamma \geq 0$ simplifies to the trivial inequality $0 \geq 0$, which gives us a contradiction. Thus, we conclude that $c_{p,q}$ is nonzero, whenever both $a_p$ and $b_q$ are nonzero, implying $Q_c = Q_{a,b}$.

Therefore, without loss of generality, we can assume that $\mu_p$ and $\lambda_q$ are nonzero for all nonzero $c_{p,q}$ as defined in (17). As a result, we can factorize $\mu_p$ as $\mu_p = \nu_{p,q} a_p$ for some nonzero $\nu_{p,q}$. By (17), it follows that

$$c_{p,q} = \nu_{p,q} a_p b_q, \quad \forall p \cup q \in Q_c.$$

(18)

Finally, consider two elements in $Q_c$ of the form $p \cup \tilde{q}$ and $p \cup \hat{q}$. Using relations (17) and (18) for $c_{p,\tilde{q}}$ and $c_{p,\hat{q}}$ yields $\mu_p = \nu_{p,\tilde{q}} a_p = \nu_{p,\hat{q}} a_p$. Therefore, $\nu_{p,q} = \nu$ for all $p \cup q \in Q_c$; i.e., $cz + \gamma \geq 0$ coincides with inequality (12) up to a scaling. In addition, since both inequalities are valid for $\text{MP}_G$, it follows that $\nu$ is positive. This in turn implies (12) defines a facet of $\text{MP}_G$. Hence, by Proposition 6, inequality (12) is facet-defining for $\text{MP}_G$.

By a recursive application of Theorem 3, we can construct certain facets of $\text{MP}_G$ from the facets of $k$ simpler polytopes $\text{MP}_{G_i}, i \in \{1, \ldots, k\}$. Next, we demonstrate the applicability of the above lifting operation via a simple example.

**Example 4.** Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_4, v_5\},$$

$$\{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$$
See Figure 4. Define a partition of the nodes of $G$ as $V = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$. Define the two hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as in Theorem 3; i.e., $E_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$ and $E_2 = \{\{v_4, v_5\}\}$. Now consider the facets of $MP_{G_1}$ and $MP_{G_2}$ given by $z_{12} + z_{13} \leq z_1 + z_{23}$ and $z_{45} \geq 0$, respectively. Then, by Theorem 3, the inequality

$$z_{1245} + z_{1345} \leq z_{145} + z_{2345}$$

is facet-defining for $MP_G$. ◊

We should remark that the converse of Theorem 3 does not hold in the following sense: let $V(G) = V_1 \cup V_2$ be any partition of the nodes of the hypergraph $G$ and let $G_1$ and $G_2$ be the corresponding hypergraphs as defined in Theorem 3. Suppose that the inequality $dz + \delta \geq 0$ defines a facet of $MP_G$ and can be obtained by linearizing $(az + \alpha)(bz + \beta) \geq 0$, where $az + \alpha \geq 0$ and $bz + \beta \geq 0$ are valid inequalities for $MP_{G_1}$ and $MP_{G_2}$, respectively. Then, these inequalities are not necessarily facet-defining for the corresponding polytopes. In addition, it might not be possible to obtain $dz + \delta \geq 0$ by multiplying and linearizing two (other) facets of $MP_{G_1}$ and $MP_{G_2}$. We demonstrate this fact via a simple example:

**Example 5.** Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2, v_3\}\}$ and consider a facet of $MP_G$ given by $z_1 - z_{123} \geq 0$. Define a partition of the nodes of $G$ as $V = V_1 \cup V_2$ with $V_1 = \{v_1\}$ and $V_2 = \{v_2, v_3\}$. Construct the two hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as defined in Theorem 3; i.e., $E_1 = \{\{v_1\}\}$ and $E_2 = \{\{v_2, v_3\}\}$. The inequality $z_1 - z_{123} \geq 0$ can be obtained by linearizing the relation $z_1(1 - z_{23}) \geq 0$. While $z_1 \geq 0$ defines a facet of $MP_{G_1}$, the inequality $1 - z_{23} \geq 0$ is not facet-defining for $MP_{G_2}$ as it is implied by $z_2 - z_{23} \geq 0$ and $1 - z_2 \geq 0$, both of which are valid inequalities for $MP_{G_2}$. It is simple to check that $z_1 - z_{123} \geq 0$ cannot be obtained by multiplying a linearizing any two facets of $MP_{G_1}$ and $MP_{G_2}$. In addition, it can be verified that there exists no partition of the nodes of $G$ that can be utilized along with Theorem 3 to generate the facet-defining inequality $z_1 - z_{123} \geq 0$. ◊

Now suppose that the hypergraph $G = (V, E)$ defined in Theorem 3 is a rank-$(r + 1)$ full hypergraph $K^{n,r+1}$. Let $V_1 = \{\hat{v}\}$ for some $\hat{v} \in V$ and let $V_2 = V \setminus \{\hat{v}\}$. Define $G_1$ to be the graph corresponding to the node $\hat{v}$ and $G_2$ to be the rank-$r$ full hypergraph with the node set $V_2$. Then clearly all assumptions of Theorem 3 are satisfied and linearizations of

$$z_0(bz + \beta) \geq 0$$

and

$$(1 - z_0)(bz + \beta) \geq 0$$

define facets of $MP^{n,r+1}$ for any facet $bz + \beta \geq 0$ of $G_2$.

More generally, by defining $G$ to be a rank-$(r + \delta)$ full hypergraph for some $\delta \geq 1$, $G_1$ to be a complete hypergraph with $\delta$ nodes, $G_2$ to be a rank-$r$ full hypergraph and utilizing Corollary 3, we obtain:

**Corollary 7.** Let $bz + \beta \geq 0$ denote a facet-defining inequality for $MP^{n,r}$. Let $W$ denote a subset of nodes of $K^{n,r}$ of cardinality $\delta \geq 1$ such that $W \cap V(G(\hat{v})) = \emptyset$. For any $U \subseteq W$, denote by $\psi_U$ the switching operator as defined by relations (3) and (4). Then the linearization of any relation of the form

$$\psi_U(z_W)(bz + \beta) \geq 0,$$

defines a facet of $MP^{n,r'}$, where $r' = r + \delta$. 
The above result provides a systematic procedure to construct facets for the convex hull of nonconvex sets containing higher degree multilines from the facets of those containing lower degree multilinear terms. For instance, various classes of facet-defining inequalities for the Boolean Quadric Polytope have been identified in the literature (c.f. [28, 11]). The result of Corollary 7 enables us to convert these facets into facets of the convex hull of the sets containing higher degree multilines, as demonstrated by the following example.

**Example 6.** Consider the Boolean Quadric Polytope $QP^n$. It is well-known that the triangle inequalities defined as

\[
z_{ij} + z_{ik} \leq z_i + z_{jk}, \\
z_i + z_j + z_k - z_{ij} - z_{ik} - z_{jk} \leq 1,
\]

for all distinct $i, j, k \in \{1, \ldots, n\}$, are facet-defining for $QP^n$ (c.f. [28]). Then, by Corollary 7, the following inequalities obtained by multiplying the triangle inequalities by $z_i$ and $1 - z_i$, $l \in \{1, \ldots, n\}$, and linearizing the resulting relations, define facets of $MP^{n,3}$:

\[
z_{ijl} + z_{ikl} \leq z_{il} + z_{jkl}, \\
z_{ij} + z_{ik} + z_{il} + z_{jkl} \leq z_i + z_{jk} + z_{ijl} + z_{ikl}, \\
z_{il} + z_{jkl} - z_{ijl} - z_{jkl} - z_{jkl} \leq z_l, \\
z_i + z_j + z_k + z_{ij} - z_{ik} - z_{jk} - z_{il} - z_{jkl} + z_{ijl} + z_{ikl} + z_{jkl} \leq 1,
\]

for all distinct $i, j, k, l \in \{1, \ldots, n\}$. \(\diamondsuit\)

Denote by $G_1$ and $G_2$ two hypergraphs with $V(G_1) \cap V(G_2) = \emptyset$, and suppose that $MP_{G_1} = \{z : a^t z + \alpha_i \geq 0, \forall i \in I\}$ and $MP_{G_2} = \{z : b^t z + \beta_j \geq 0, \forall j \in J\}$. We define the multiplication hypergraph $G_1 \times G_2$ of $G_1$ and $G_2$ as the hypergraph with node set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{p \cup q : p \in L(G_1) \cup E(G_1), q \in L(G_2) \cup E(G_2)\}$. Let the polytope $P_{G_1 \times G_2}$ be defined by the linearization of every relation of the form $(a^t z + \alpha_i)(b^t z + \beta_j) \geq 0$, for all $i \in I$ and $j \in J$. It is simple to verify that the polytope $P_{G_1 \times G_2}$ is well-defined. Namely, $P_{G_1 \times G_2}$ remains unchanged if any number of redundant inequalities are added to the descriptions of $MP_{G_1}$ and $MP_{G_2}$. Clearly, $P_{G_1 \times G_2} \supseteq MP_{G_1 \times G_2}$. We are interested in identifying the cases for which $MP_{G_1 \times G_2} = P_{G_1 \times G_2}$. Observe that for any hypergraph $G$ different from $G_1 \times G_2$ and containing $G_1 \times G_2$ as a subhypergraph, we have $P_{G_1 \times G_2} \supseteq MP_G$, since any edge $e \in E(G) \setminus E(G_1 \times G_2)$ does not belong to the support hypergraph of any inequality defining $P_{G_1 \times G_2}$. In the following, we investigate the relationship between the two polytopes $P_{G_1 \times G_2}$ and $MP_{G_1 \times G_2}$.

The next theorem shows that if one of the two hypergraphs, say $G_2$, is a single node, then $P_{G_1 \times G_2} = MP_{G_1 \times G_2}$. The proof technique used in Theorem 4 is similar to the disjunctive programming approach of Balas [2] who gives an extended formulation for the convex hull of the union of finitely many polyhedra. In our case we are able to explicitly project such formulation, and characterize the convex hull in the space of the original variables.

**Theorem 4.** Let $G_1$ be a hypergraph with $MP_{G_1} = \{z : a^t z + \alpha_i \geq 0, \forall i \in I\}$, and let $G_2$ be the graph corresponding to a single node $\tilde{v} \notin V(G_1)$. Then the polytope $MP_{G_1 \times G_2}$ is defined by the linearization of the following relations:

\[
z_{\tilde{v}}(a^t z + \alpha_i) \geq 0, \quad (1 - z_{\tilde{v}})(a^t z + \alpha_i) \geq 0,
\]

for all $i \in I$.

**Proof.** Denote by $\tilde{G}$ the multiplication hypergraph $G_1 \times G_2$. Consider the faces of $MP_{\tilde{G}}$ given by $F^0 = \{z \in MP_{\tilde{G}} : z_{\tilde{v}} = 0\}$ and $F^1 = \{z \in MP_{\tilde{G}} : z_{\tilde{v}} = 1\}$. From the definition of $MP_{\tilde{G}}$ it follows that

\[
F^0 = \{z : z_{\tilde{v}} = 0, \, z_{p \cup \{\tilde{v}\}} = 0, \, z_p \in MP_{G_1}, \, \forall p \in L(G_1) \cup E(G_1)\},
\]

and
and \[ F^1 = \{ z : z_0 = 1, z_{p \cup \{ i \}} = z_p, z_p \in MP_{G_i}, \forall p \in L(G_1) \cup E(G_1) \}. \]

Since MP\(_G\) is an integral polytope, any point \( z \in MP_G \) can be written as \( z = (1 - \lambda)z^0 + \lambda z^1 \) for some \( 0 \leq \lambda \leq 1 \), where \( z^0 \in F^0 \) and \( z^1 \in F^1 \). Thus, \( MP_G \) can be equivalently written as:

\[
MP_G = \left\{ z : z_0 = \lambda, z_p = (1 - \lambda)z_p^0 + \lambda z_p^1, z_{p \cup \{ i \}} = \lambda z_p^1, \forall p \in L(G_1) \cup E(G_1), \right. \\
\left. z^0, z^1 \in MP_{G_i}, 0 \leq \lambda \leq 1 \right\}.
\]

(19)

Our objective is to derive an explicit description for \( MP_G \) in the space of \( z \) variables. We start by eliminating \( \lambda \) from the description of \( MP_G \) using \( z_0 = \lambda \).

Note that if \( z_0 = 0 \) (i.e., \( \lambda = 0 \)), then \( z^1 \in MP_{G_i} \) is redundant, and if \( z_0 > 0 \), then \( z^1 \in MP_{G_i} \) is equivalent to

\[ z_0(a^i z^1 + \alpha_i) \geq 0, \quad \forall i \in I. \]

(20)

Therefore, the constraint \( z^1 \in MP_{G_i} \) in (19) can be replaced with (20). Similarly, the constraint \( z^0 \in MP_{G_i} \) in (19) can be written as

\[ (1 - z_0)(a^i z^0 + \alpha_i) \geq 0, \quad \forall i \in I. \]

(21)

By \( z_{p \cup \{ i \}} = z_0 z_p^1 \), inequalities (20) can be equivalently written as \( \sum_p a_p^i z_{p \cup \{ i \}} + \alpha_i z_0 \geq 0 \) for all \( i \in I \), where \( p \in L(G_1) \cup E(G_1) \), which is identical to the linearization of \( z_0(a^i z + \alpha_i) \geq 0 \) for all \( i \in I \). In addition, from \( z_{p \cup \{ i \}} = z_0 z_p^1 \) and \( z_p = (1 - z_0)z_p^0 + z_0 z_p^1 \) for all \( p \in L(G_1) \cup E(G_1) \), it follows that \( (1 - z_0)z_p^0 = z_p - z_{p \cup \{ i \}} \). Hence, inequalities (21) are equivalently given by \( \sum_p a_p^i (z_p - z_{p \cup \{ i \}}) + (1 - z_0)\alpha_i \geq 0 \) for all \( i \in I \), and the latter system of inequalities is identical to the linearization of the system \( (1 - z_0)(a^i z + \alpha_i) \geq 0 \) for all \( i \in I \).

More generally, let \( G_2 \) be a complete hypergraph. Then, by a repeated application of Theorem 4, we conclude that also in this case \( \mathcal{P}_{G_1 \times G_2} = MP_{G_1 \times G_2} \):

**Corollary 8.** Let \( G_1 \) be a hypergraph with \( MP_{G_1} = \{ z : a^i z + \alpha_i \geq 0, \forall i \in I \} \), and let \( G_2 \) be a complete hypergraph with \( V(G_1) \cap V(G_2) = \emptyset \). Then the polytope \( MP_{G_1 \times G_2} \) is defined by the linearization of the following relations:

\[ \psi_U(z_{V(G_1)})(a^i z + \alpha_i) \geq 0, \quad \forall i \in I, \forall U \subseteq V(G_1). \]

However, as we demonstrate in the following examples, if the hypergraphs \( G_1 \) and \( G_2 \) are both not complete, then \( MP_{G_1 \times G_2} \) is strictly contained in \( \mathcal{P}_{G_1 \times G_2} \), in general.

**Example 7.** Consider the two graphs \( G_1 \) and \( G_2 \) with \( V(G_1) = \{ v_1, v_2 \}, E(G_1) = \emptyset \), \( V(G_2) = \{ v_3, v_4 \} \) and \( E(G_2) = \emptyset \). It follows that \( MP_{G_1} = \{ z : 0 \leq z_i \leq 1, i = 1, 2 \} \), \( MP_{G_2} = \{ z : 0 \leq z_i \leq 1, i = 3, 4 \} \) and \( \mathcal{P}_{G_1 \times G_2} = \{ z : z_{ij} \geq 0, z_i - z_j \geq 0, z_j - z_i \geq 0, z_{ij} - z_i - z_j + 1 \geq 0, (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\} \} \). However, in this case, the multiplication hypergraph \( G_1 \times G_2 \) consists of a cordless cycle of length four; i.e., \( V(G_1 \times G_2) = \{ v_1, v_2, v_3, v_4 \} \) and \( E(G_1 \times G_2) = \{ \{ v_1, v_3 \}, \{ v_1, v_4 \}, \{ v_2, v_3 \}, \{ v_2, v_4 \} \} \). It is well known that an inequality of the form \( z_{13} + z_{14} + z_{23} \leq z_{24} + z_{13} + z_{3} \) defines a facet of \( MP_{G_1 \times G_2} \) (c.f. [28]), which is clearly not included in the description of \( \mathcal{P}_{G_1 \times G_2} \). Thus, in this example \( \mathcal{P}_{G_1 \times G_2} \subset MP_{G_1 \times G_2} \).

In Example 7, both \( G_1 \) and \( G_2 \) are disconnected graphs. One might wonder if \( MP_{G_1 \times G_2} = \mathcal{P}_{G_1 \times G_2} \) holds for any two connected hypergraphs \( G_1 \) and \( G_2 \). The following example shows that such a claim is not valid.
**Example 8.** Consider the two hypergraphs $G_1$ and $G_2$ with $V(G_1) = \{v_1, v_2, v_3\}$, $E(G_1) = \{(v_1, v_2, v_3)\}$, $V(G_2) = \{v_4, v_5, v_6\}$, and $E(G_2) = \{(v_4, v_5, v_6)\}$. Consider the multiplication hypergraph $G_1 \times G_2$ as defined above. It can be shown that an inequality of the form $-z_1 - z_2 + z_3 - z_4 + z_5 - z_6 \leq 0$ defines a facet of $MP_{G_1 \times G_2}$. However, it is simple to check that this inequality cannot be obtained by multiplying and linearizing any two facets of $MP_{G_1}$ and $MP_{G_2}$.

Next, we utilize Theorem 4 to investigate the converse of the result considered in Corollary 7. More precisely, consider the polytope $MP^{n,r}$, $r \geq 2$. For each facet-defining inequality $az + \alpha \geq 0$, denote by $U \subset V(K^{n,r})$ the set of nodes that are not present in $G(a)$. Then, by Corollary 7, multiplying $az + \alpha \geq 0$ by $z_v$ (or $1 - z_v$) for each $v \in U$, and linearizing the resulting inequality gives a facet of $MP^{n,r+1}$. Denote by $P^{n,r+1}$ the polytope defined by all such inequalities. We would like to characterize the relationship between $MP^{n,r+1}$ and $P^{n,r+1}$. Such a result is of particular interest, as for instance it enables us to identify the structure of those facets of $MP^{n,r+1}$, $r \geq 2$ that cannot be obtained by lifting facets of the Boolean Quadratic Polytope and as a result require different derivation techniques.

Before addressing the above question, we should remark that to construct the polytope $P^{n,r+1}$, we multiply each facet of $MP^{n,r}$ by those variables that are not present in the support hypergraph of the corresponding facet. Consider a facet of $MP^{3,2}$ defined by $z_1 + z_2 \leq z_1 + z_2$. Multiplying this facet by $z_3$ and linearizing the resulting inequality yields $z_{123} \leq z_{23}$, which indeed defines a facet of $MP^{3,3}$. However, the same facet can also be obtained by considering another facet of $MP^{3,2}$ defined by $z_{12} \leq z_2$, multiplying this facet by $z_3$ and linearizing the resulting relation. In fact, the result of Theorem 4 implies that in general, it suffices to consider the nodes that are not present in the support hypergraphs of the facets of $MP^{n,r}$. To see this, consider a facet of $MP^{n,r}$ given by $az + \alpha \geq 0$ and let $\bar{v}$ denote a node that belongs to the hypergraph $G(a)$, such that the linearization of $z_{\bar{v}}(az + \alpha) \geq 0$ (or $(1 - z_{\bar{v}})(az + \alpha) \geq 0$) defines a facet of $MP^{n,r+1}$. Clearly, the support hypergraph of the new facet is a subhypergraph of the hypergraph $G_1 \times G_2$, where $G_1$ is the rank-$r$ full hypergraph on the nodes different from $\bar{v}$, and $G_2$ is the graph corresponding to the single node $\bar{v}$. Therefore, by Theorem 4, the same facet can be obtained by multiplying a facet of $MP^{n-1,r}$ by $z_{\bar{v}}$ (or $(1 - z_{\bar{v}})$), and subsequently linearizing it.

Now let us return to the question of the relationship between the two polytopes $MP^{n,r+1}$ and $P^{n,r+1}$, for $r \geq 2$. Let $G_1 = K^{n-1,r}$, and let $G_2$ be a graph corresponding to a single node not in $V(G_1)$. In this case, $G_1 \times G_2$ is a subhypergraph of $K^{n,r+1}$, and in fact, the missing edges are precisely those rank-$(r+1)$ edges of $K^{n,r+1}$ contained in $V(G_1)$. It then follows by Theorem 4 that the polytope $P^{n,r+1}$ contains any facet of $MP^{n,r+1}$ whose support hypergraph is a subhypergraph of $G_1 \times G_2$. In particular, if $n = r + 1$, then we have $MP^{n,n} = P^{n,n}$. For the general case with $n > r + 1$, by Theorem 4, we can state the following result:

**Corollary 9.** Let $az + \alpha \geq 0$ denote a facet of $MP^{n,r+1}$ and denote by $\hat{E}$ the set of all rank-$(r+1)$ edges in $G(a)$. Then $az + \alpha \geq 0$ can be obtained by linearizing a relation of the form $z_{\hat{v}}(bz + \beta) \geq 0$ or of the form $(1 - z_{\hat{v}})(bz + \beta) \geq 0$, where $bz + \beta \geq 0$ defines a facet of $MP^{n-1,r}$ and $\hat{v} \notin V(G(b))$, if and only if $\hat{v} \in \cap_{e \in \hat{E}} E^e$.

**Proof.** We first prove sufficiency of the condition. Let $az + \alpha \geq 0$ be obtained by linearizing a relation of the form $z_{\hat{v}}(bz + \beta) \geq 0$ or of the form $(1 - z_{\hat{v}})(bz + \beta) \geq 0$, where $bz + \beta \geq 0$ defines a facet of $MP^{n-1,r}$ and $\hat{v} \notin V(G(b))$. Then clearly all the edges of $G(a)$ of rank $(r+1)$ contain the node $\hat{v}$.

Let $\bar{v} \in \cap_{e \in \hat{E}} E^e$. Then necessity of the condition follows by applying Theorem 4 to the rank-$r$ full hypergraph $G$ constructed on the $n-1$ nodes different from $\hat{v}$. \qed
5 Lifting via node addition

In this section, we introduce a different lifting operation in which the set $S_G'$ is obtained by fixing certain independent variables in $S_G$ to one; that is, we set $z_0 = 1$ for some $v \in V(G)$. Equivalently, the hypergraph $G'$ is obtained from the hypergraph $G$ by removing certain nodes of $G$. More precisely, given a node $\bar{v} \in V(G)$, we say that $G'$ is obtained from $G$ by removing $\bar{v}$, if $V(G') = V(G) \setminus \{\bar{v}\}$ and $E(G') = \{e \setminus \{\bar{v}\} : e \in E(G), |e \setminus \{\bar{v}\}| \geq 2\}$. This type of lifting can be used to obtain facets of sets containing higher degree multilinear polynomials from those with lower order ones.

As we detail in the following, our results are based on the key assumption that $\dim(MP_G) = \dim(MP_G') + 1$, and this relation holds if and only if the hypergraph $G'$ does not contain any loops or parallel edges; i.e., $e_j \setminus \{\bar{v}\} \notin L(G) \cup E(G)$ for all $j \in J$, where $J$ denotes the index set of edges of $G$ containing $\bar{v}$. This assumption is needed as otherwise the polytope $MP_G'$ is not full-dimensional. It then follows that there exist linearly independent inequalities defining the same facet of $MP_G'$, in which case the lifting operations of this section are not well-defined.

**Theorem 5.** Let $G$ be a hypergraph, let $\bar{v} \in V(G)$, and let $\{\bar{e}_j : j \in J\}$ be the set of all edges containing $\bar{v}$. Let $e_j = \bar{e}_j \setminus \{\bar{v}\}$ for each $j \in J$ and suppose that $e_j \notin L(G) \cup E(G)$ for all $j \in J$. Let $G'$ be the hypergraph obtained from $G$ by removing the node $\bar{v}$. Denote by $az \leq \alpha$ a valid inequality for $MP_G'$. Define $J = \{j \in J : a_{e_j} \neq 0\}$. Let $\bar{v} \in V(G')$ with $a_{\bar{v}} \geq 0$. Let $\{e_j : j \in J\}$ be the set of all edges in $G(a)$ that contain $\bar{v}$, and suppose that $J = \tilde{J}$. Then the inequality

$$\sum_{p \in L(G') \cup E(G') \setminus \{e_j : j \in J\}} a_p \bar{z}_p + \sum_{j \in \tilde{J}} a_{e_j} \bar{z}_{e_j} + a_{\bar{v}} \bar{z}_{\bar{v}} \leq \alpha + a_{\bar{v}}$$

(22)

is valid for $MP_G$. Moreover, if $az \leq \alpha$ is facet-defining for $MP_G'$ and is different from $z_0 \leq 1$, then (22) is facet-defining for $MP_G$.

**Proof.** We start by establishing the validity of inequality (22) for every point $\bar{z} \in S_G$. If $\bar{z}_e = 1$, then the validity of (22) follows from $\bar{z}_{e_j} = \bar{z}_{e_j}$ for all $j \in J$. Hence, let $\bar{z}_{\bar{v}} = 0$. In this case, if $\bar{z}_{\bar{v}} = 1$, then the validity of (22) follows from the previous argument, the symmetry of the support hypergraph of inequality (22) with respect to $\bar{v}$ and $\bar{v}$, and the fact that the coefficients of $z_{\bar{v}}$ and $z_0$ in (22) are identical. Thus, it suffices to show the validity of (22) if $\bar{z}_{\bar{v}} = \bar{z}_0 = 0$. Let $\bar{z} \in S_G'$ be obtained by dropping $z_0$ and computing $\hat{y}_e$ accordingly for every $e \in E(G')$. It then follows that the value of the left hand side of inequality (22) at $\bar{z}$ is equal to the value of the $\bar{v}$ hand side of $az \leq \alpha$ at $\bar{z}$. By assumption $a_{\bar{v}} \geq 0$, hence inequality (22) is valid for $MP_G$.

We now show that if $az \leq \alpha$ is facet-defining for $MP_G'$, then inequality (22) defines a facet of $MP_G$. By assumption $e_j \notin L(G) \cup E(G)$ for all $j \in J$, implying $\dim(MP_G) = \dim(MP_G') + 1$. Denote by $z^i$, $i = 1, \ldots, k$, all points in $S_{G'}$ satisfying $az = \alpha$. We lift each of these points $z^i$ to a point $\bar{z}^i$ in $S_G$ by letting $\bar{z}_{e_j} = 1$, and by computing $\bar{z}_{e_j}$ accordingly for every $e \in E(G)$. Clearly, $\bar{z}_{e_j} = z_{e_j}$, for all $j \in J$, implying inequality (22) is satisfied tightly at these points. Since $az \leq \alpha$ defines a facet of $MP_G'$, there are at least $|V(G')| + |E(G')|$ affinely independent points among $\bar{z}^i$, $i = 1, \ldots, k$.

To complete the proof, we need one additional point in $S_G$ denoted by $\hat{z}$ that (i) cannot be written as an affine combination of $\bar{z}^i$, $i = 1, \ldots, k$, and (ii) satisfies (22) tightly. Clearly, any point with $\hat{z}_{\bar{v}} = 0$ satisfies condition (i), since $\bar{z}^i = z_0$ for all $i = 1, \ldots, k$. We choose a point in $\bar{z}^0 \in S_{G'}$ with $az^0 = \alpha$, and $z_0^0 = 0$. The existence of such a point follows from the assumption that the facet-defining inequality $az \leq \alpha$ is different from $z_0 \leq 1$. Next, we lift $\hat{z}$ to a point $\hat{z}$ in $S_G$ by letting $\hat{z}_{e_j} = 1$, $\hat{z}_{\bar{v}} = 0$, and by computing $\hat{z}_{e_j}$ accordingly for every $e \in E(G)$. Note that $\hat{z}_{e_j} = z_{e_j}^0 = 0$ for every $j \in J = \tilde{J}$. Therefore, $\hat{z}$ satisfies (22) tightly and as a result, inequality (22) is facet-defining for $MP_G$. □
The key assumption in Theorem 5 is the symmetric structure of the support hypergraph of inequality (22) with respect to the nodes \( \bar{v} \) and \( \bar{v} \). Clearly, this assumption is satisfied in the special case where the hypergraph \( G \) is symmetric with respect to \( \bar{v} \) and \( \bar{v} \); i.e., the two nodes are contained in the same set of edges of \( G \). In the following example, we demonstrate the applicability of the above lifting operation.

**Example 9.** Consider the hypergraph \( G \) defined as \( V(G) = \{v_1, v_2, v_3, v_4, v_5\} \) and \( E(G) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\} \) (see Figure 5). We show that the following inequalities are facet-defining for \( MP_G \):

\[
\begin{align*}
-z_1 + z_{13} + z_{14} - z_{345} & \leq 0, \\
-z_3 + z_{13} - z_{145} + z_{345} & \leq 0, \\
z_1 + z_3 + z_4 + z_5 - z_{13} - z_{145} - z_{345} & \leq 2.
\end{align*}
\]

(23)

To see this, consider the graph \( G' \) obtained by removing the node \( \bar{v} = v_5 \) from \( G \); i.e., \( V(G') = \{v_1, v_2, v_3, v_4\} \) and \( E(G') = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\} \). First, note that the set of edges of \( G' \) corresponding to edges in \( G \) containing \( v_5 \) is \( E_J = \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\} \). Moreover, it is simple to verify that the following triangle inequalities are facet-defining for \( MP_{G'} \):

\[
\begin{align*}
-z_1 + z_{13} + z_{14} - z_{34} & \leq 0, \\
-z_3 + z_{13} - z_{14} + z_{34} & \leq 0, \\
z_1 + z_3 + z_4 + z_{13} - z_{14} - z_{34} & \leq 1.
\end{align*}
\]

(24)

The set of edges in \( E_J \) with nonzero coefficients in each of the above inequalities is \( E_J = \{\{v_1, v_4\}, \{v_3, v_4\}\} \). Now let \( \bar{v} = v_4 \). Clearly, in all three inequalities (24), we have \( \alpha_{\bar{v}} \geq 0 \). Moreover, for all these inequalities the set of edges with nonzero coefficients containing \( v_4 \) is given by \( E_{\bar{v}} = \{\{v_1, v_4\}, \{v_3, v_4\}\} \). It follows that \( E_J = E_{\bar{v}} \). Therefore, all assumptions of Theorem 5 are satisfied and inequalities (23) are facet-defining for \( MP_{G'} \).

Next, we develop alternative lifting operations for cases that do not satisfy the assumptions of Theorem 5. We make use of the following proposition to present our next result.

**Proposition 8.** Let \( G \) be a hypergraph, and let \( az \leq \alpha \) be a facet-defining inequality for \( MP_G \). Let \( v \in V(G(a)) \), and let \( e_j, j \in J \), be the edges of \( G(a) \) that contain \( v \). Suppose that \( az \leq \alpha \) is different from \( z_v \leq 1 \). If \( \sum_{j \in J} a_{e_j} z_{e_j} \geq 0 \) for every \( z \in S_G \), then \( a_v \leq 0 \).

**Proof.** Since \( az \leq \alpha \) is different from \( z_v \leq 1 \), it follows that there exists \( \bar{z} \in S_G \) with \( a\bar{z} = \alpha \) and \( \bar{z}_v = 0 \) (and \( \bar{z}_{e_j} = 0 \) for every \( j \in J \)). Let \( P = L(G) \cup E(G) \setminus \{\{v\}, e_j : j \in J\} \). We have

\[
\sum_{p \in P} a_p \bar{z}_p = \alpha.
\]

(25)
Consider now the point \( \bar{z} \) obtained from \( \bar{v} \) by setting \( \bar{z}_v = 1 \) and computing the corresponding \( \bar{z}_e \) for every \( e \in E(G) \). As \( az \leq \alpha \) is valid for \( \bar{z} \in S_G \), we have \( \sum_{p \in P} a_p \bar{z}_p + a_v + \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \leq \alpha \). Since \( \bar{z}_p = \bar{z}_p \) for every \( p \in P \), we obtain

\[
\sum_{p \in P} a_p \bar{z}_p + a_v + \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \leq \alpha. \tag{26}
\]

From (25) and (26) we get \( a_v \leq -\sum_{j \in J} a_{e_j} \bar{z}_{e_j} \). As \( \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \geq 0 \) for every \( z \in S_G \), we conclude that \( a_v \leq 0 \).

Consider the hypergraphs \( G \) and \( G' \) defined in Theorem 5 and let \( az \leq \alpha \) be facet-defining for \( MP_{G'} \). In the next theorem, we introduce a lifting operation assuming that \( \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \geq 0 \) for every \( z \in S_{G'} \), where \( J \) corresponds to the index set of edges in \( G(a) \) containing \( \bar{v} \). In this case, by Proposition 8 we have \( a_\bar{v} \leq 0 \). Clearly, if \( a_\bar{v} < 0 \), then the lifting technique of Theorem 5 cannot be utilized. The following lifting operation is applicable in many cases for which Theorem 5 cannot be applied.

**Theorem 6.** Let \( G \) be a hypergraph, let \( \bar{v} \in V(G) \), and let \( G' \) be obtained from \( G \) by removing \( \bar{v} \). Denote by \( \{ \bar{e}_j : j \in J \} \) the set of all edges containing \( \bar{v} \). Suppose that \( \bar{e}_j \setminus \{ \bar{v} \} \notin L(G) \cup E(G) \) for all \( j \in J \). Let \( az \leq \alpha \) denote a valid inequality for \( MP_{G'} \) such that

\[
\sum_{j \in J} a_{e_j} \bar{z}_{e_j} \geq 0, \quad \forall z \in S_{G'}, \tag{27}
\]

where \( e_j = \bar{e}_j \setminus \{ \bar{v} \} \) for each \( j \in J \). Then, the inequality

\[
\sum_{e \in L(G') \cup E(G') \setminus \{ \bar{e}_j : j \in J \}} a_e \bar{z}_e + \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \leq \alpha, \tag{28}
\]

is valid for \( MP_G \). Denote by \( \bar{J} = \{ j \in J : a_{e_j} \neq 0 \} \) and suppose that

\[
\bigcap_{j \in \bar{J}} \bar{e}_j \neq \emptyset. \tag{29}
\]

If \( az \leq \alpha \) is facet-defining for \( MP_{G'} \), then inequality (28) is facet-defining for \( MP_G \).

**Proof.** We start by establishing the validity of inequality (28) for \( MP_G \). Let \( \bar{z} \) be a feasible point in \( S_G \), and let \( \bar{z}_e \) be the corresponding point in \( S_{G'} \), obtained by dropping \( \bar{z}_v \) and computing the corresponding feasible components \( \bar{z}_e \), for all \( e \in E(G') \). Two cases arise:

(i) \( \bar{z}_e = 1 \); it then follows that \( \bar{z}_{\bar{e}_j} = \bar{z}_{e_j} \) for all \( j \in J \) which in turn implies inequality (28) is valid at \( \bar{z} \).

(ii) \( \bar{z}_e = 0 \); in this case, substituting \( \bar{z} \) in inequality (28) yields

\[
\sum_{e \in L(G') \cup E(G') \setminus \{ \bar{e}_j : j \in J \}} a_e \bar{z}_e \leq \alpha \tag{30}
\]

By assumption, we have \( \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \geq 0 \). From the relation \( a \bar{z} \leq \alpha \), it then follows that \( \sum_{e \in L(G') \cup E(G') \setminus \{ \bar{e}_j : j \in J \}} a_e \bar{z}_e \leq \alpha \). Moreover, \( \bar{z}_e = \bar{z}_e \) for all \( e \in L(G') \cup E(G') \setminus \{ e_j : j \in J \} \). Hence, inequality (30) is valid.
We now show that if $az \leq \alpha$ is facet-defining for $MP_{G'}$ and condition (29) is satisfied, then inequality (28) defines a facet of $MP_G$. Denote by $z^i$, $i = 1, \ldots, k$ the set of all points in $S_{G'}$ satisfying $az = \alpha$. We lift each of these points $z^i$ to a point $\hat{z}^i$ in $S_{G}$ by letting $\hat{z}^i_0 = 1$, and by computing $\hat{z}^i_e$ accordingly, for each $e \in E(G)$. Clearly, $\hat{z}^i_j = z^i_{e_j}$, for all $j \in J$. Hence, inequality (28) is satisfied tightly at these points. Since $az \leq \alpha$ is facet-defining for $MP_{G'}$, the set $\{\hat{z}^i : i = 1, \ldots, k\}$ contains $|V(G')| + |E(G')|$ affinely independent points.

By assumption, $e_j \notin L(G) \cup E(G)$ for all $j \in J$. It follows that $\dim(MP_G) = \dim(MP_{G'}) + 1$. Consequently, to complete the proof, we need one additional point in $S_G$ denoted by $\bar{z}$ that satisfies (28) tightly and cannot be written as an affine combination of the points $\hat{z}^i$, $i = 1, \ldots, k$. Clearly, any point $\bar{z}$ with $\bar{z}_0 = 0$ cannot be written as an affine combination of $\hat{z}^i$, $i = 1, \ldots, k$, since $\bar{z}^i_0 = 1$ for all $i = 1, \ldots, k$.

We first show that there exists a point $z^0 \in S_{G'}$ with $az^0 = \alpha$, and $z^0_{e_j} = 0$ for every $j \in \bar{J}$. Let $\bar{v} \in \cap_{j \in \bar{J}} e_j$. Note that by (29), a node $\bar{v}$ always exists. Two cases arise: (i) if $az \leq \alpha$ is different from $z_{\bar{v}} \leq 1$, then there exists $z^0 \in S_{G'}$ with $az^0 = \alpha$ and $z^0_{e_j} = 0$ for every $j \in \bar{J}$, (ii) if $az \leq \alpha$ coincides with $z_{\bar{v}} \leq 1$, then let $z^0$ be any point in $S_{G'}$ with $z^0_{e_j} = 1$ and, for every $j \in \bar{J}$, $z^0_{e_j} = 0$ for a node $v_j \in e_j \backslash \{\bar{v}\}$, which in turn implies $z^0_{e_j} = 0$ for every $j \in \bar{J}$. Next, we lift $z^0$ to a point $\hat{z}$ in $S_G$ by letting $\hat{z}_0 = 0$, and by computing $\hat{z}_e$ accordingly, for each $e \in E$. Note that, since $z^0_{e_j} = 0$ for all $j \in \bar{J}$, it follows that $\hat{z}_{e_j} = z^0_{e_j}$ for every $j \in \bar{J}$, and therefore $\hat{z}$ satisfies (28) tightly. Thus, inequality (28) is facet-defining for $MP_G$.

Let us consider the case for which the assumptions of both Theorems 5 and 6 are satisfied. By Proposition 8, it then follows that $a_v = 0$, implying the two lifted inequalities defined by (22) and (28) are identical. We should remark that Theorem 6 relies on the assumption that $\sum_{j \in J} a_{e_j} z_{e_j} \geq 0$ for all $z \in S_{G'}$. Clearly, this assumption is satisfied in the special case where $a_{e_j} \geq 0$ for all $j \in J$; i.e., the lifting operation of Theorem 6 can be utilized for the case in which the node to be removed is located at the intersection of edges of $G$ whose corresponding coefficients in $az \leq \alpha$ are nonnegative. However, the assumption of Theorem 6 enables us to obtain facets in cases for which the latter nonnegativity assumption is not satisfied.

In the following example, we show the usefulness of the lifting operation defined in Theorem 6 to generate certain facets of a rank-3 hypergraph by lifting facets of a rank-2 hypergraph.

**Example 10.** Consider the hypergraph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{\{v_1, v_2\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4, v_5\}\}$ (see Figure 6). We claim that the following inequalities are facet-defining for $MP_G$:

$$-z_3 - z_{12} + z_{134} + z_{2345} \leq 0, \quad -z_4 - z_{12} + z_{134} + z_{2345} \leq 0. \quad (31)$$

To see this, consider the hypergraph $\tilde{G}$ obtained by removing node $v_5$ from $G$, and the hypergraph $G'$ obtained by removing node $v_4$ from $G$. It is simple to verify that the following so-called triangle inequality

$$-z_3 - z_{12} + z_{13} + z_{23} \leq 0 \quad (32)$$

Figure 6: Hypergraphs $G$, $\tilde{G}$, and $G'$ of Example 10 demonstrating that certain facets of $MP_G$ can be obtained from those of $MP_{G'}$ by employing the lifting operation defined in Theorem 6.
defines a facet of $MP_G$. (c.f. [28]). Since the coefficients of $z_{13}$ and $z_{23}$ in inequality (32) are nonnegative, by Theorem 6 and using a symmetry argument, it follows that the following inequalities are facet-defining for $MP_G$:

$$-z_3 - z_{12} + z_{134} + z_{234} \leq 0, \quad -z_4 - z_{12} + z_{134} + z_{234} \leq 0. \quad (33)$$

Again, since the coefficient of $z_{234}$ in both inequalities defined in (33) is nonnegative, we can utilized Theorem 6 to conclude that inequalities (31) are facet-defining for $MP_G$. ⊖

It is important to note that the assumption defined by (29) requiring the existence of a node $\bar{v}$ at the intersection of certain edges of $G$ in Theorem 6 is weaker than the corresponding assumption in Theorem 5. Namely, while Theorem 6 requires the existence of $\bar{v}$ whose corresponding coefficients in $az \leq \alpha$ are nonzero, Theorem 5 requires that, in addition, the node $\bar{v}$ should not be contained in any other edge of $G(a)$. This in turn implies that inequality (28) is not necessarily symmetric with respect to $\bar{v}$ and $\tilde{v}$, whereas, inequality (22) has such a symmetric structure. In the following example, we demonstrate that Theorem 6 does not hold in general, if assumption (29) is not satisfied.

Example 11. Consider the hypergraph $G$ defined as $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E(G) = \{\{v_1, v_2\}, \{v_2, v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5, v_7\}, \{v_5, v_6\}, \{v_1, v_6, v_7\}\}$ (see Figure 7). Let $\bar{v} = v_7$. For this example, the hypergraph $G'$ obtained by removing node $v_7$ from $G$ is a cordless cycle of length six. It is well-known that the following so-called odd cycle inequality defines a facet of $MP_{G'}$ (c.f. [28]):

$$-z_{12} + z_{23} - z_{34} + z_{45} + z_{56} + z_{16} \leq 1.$$  

Since the coefficients of $z_{23}$, $z_{45}$ and $z_{16}$ in the above inequality are nonnegative, if we relax the assumption on the nonemptiness of the intersection of the corresponding edges in $G'$, by Theorem 6, one concludes that the following inequality defines a facet of $MP_G$:

$$-z_{12} + z_{237} - z_{34} + z_{457} - z_{56} + z_{167} \leq 1. \quad (34)$$

We now show that the above inequality is not facet-defining for $MP_G$ by providing a valid inequality for $MP_G$ that implies inequality (34). Consider the expression on the left hand side of inequality (34). We first compute the maximum value of this expression over $MP_G$; that is, we find the maximum of $f = -z_{12}z_2 - z_3z_4 - z_5z_6 + z_1z_6z_7 + z_2z_3z_7 + z_4z_5z_7$, where $z_v \in \{0, 1\}$ for all $v \in V(G)$. Consider the following cases:

(i) $z_7 = 0$: in this case $f$ simplifies to $-z_1z_2 - z_3z_4 - z_5z_6$ whose maximum over $\{0, 1\}^7$ is equal to zero.

(ii) $z_7 = 1$: in this case, we have $f = -z_1z_2 + z_2z_3 - z_3z_4 + z_4z_5 - z_5z_6 + z_1z_6$ and it is simple to verify that $f \leq 1.$

Figure 7: Hypergraphs $G$ and $G'$ of Example 11 demonstrating that the nonemptiness assumption defined by (29) in Theorem 6 is necessary, in general.
Thus, the following is a valid inequality for $MP_G$:

$$-z_{12} + z_{237} - z_{34} + z_{457} - z_{56} + z_{167} \leq z_7.$$ 

Clearly, the above inequality, together with $z_7 \leq 1$, implies (34) and as result inequality (34) is not facet-defining for $MP_G$. Thus, we conclude that the lifting operation of Theorem 6 is not valid in general, if the nonemptyness assumption defined by (29) does not hold. ◊

In Theorem 6, if the node $\bar{v}$ is contained in a single edge $\bar{e}$, with $|\bar{e}| \geq 3$ and $\bar{e} \notin E(G)$, then the assumptions of the theorem simplify to $a_{\bar{e}\setminus\{\bar{v}\}} \geq 0$; i.e., the node $\bar{v}$ is restricted to be removed from an edge whose corresponding coefficient in the facet of $MP_G$ is nonnegative. In the following theorem, we consider the case where $\bar{v}$ is removed from an edge with a negative coefficient.

**Theorem 7.** Let $G$ be a hypergraph and let $\bar{v}$ be a node of $G$ that is contained only in one edge $\bar{e} \in E(G)$. Suppose that $|\bar{e}| \geq 3$, and that $\bar{e} \notin E(G)$. Let $G'$ be obtained from $G$ by removing $\bar{v}$, and let $\tilde{e} = \bar{e} \setminus \{\bar{v}\}$. Let $az \leq \alpha$ denote a valid inequality for $MP_{G'}$ with $a_{\tilde{e}} < 0$. Then, the inequality

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p z_p + a_{\tilde{e}} z_{\tilde{e}} - a_{\tilde{e}} z_{\tilde{e}} \leq \alpha - a_{\tilde{e}} \tag{35}$$

is valid for $MP_G$. Moreover, if $az \leq \alpha$ is facet-defining for $MP_{G'}$ and is different from $z_{\tilde{e}} \geq 0$, then (35) is facet-defining for $MP_G$.

**Proof.** We first establish the validity of inequality (35) for $MP_G$. Let $\tilde{z}$ be a feasible point in $S_G$. We show that inequality (35) is satisfied by $\tilde{z}$. Let $\tilde{z}$ be the corresponding point in $S_{G'}$, obtained by dropping $z_{\tilde{e}}$, and by computing the corresponding feasible $\tilde{z}_{\tilde{e}}$, for $e \in E(G')$. Note that $\tilde{z}_p = \tilde{z}_{\tilde{e}}$ for every $p \in L(G') \cup E(G') \setminus \{\tilde{e}\}$.

First, let $\tilde{z}_{\tilde{e}} = 1$. In this case, the validity of inequality (35) follows from the fact that $\tilde{z}_{\tilde{e}} = \tilde{z}_{\tilde{e}}$:

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \tilde{z}_p + a_{\tilde{e}} \tilde{z}_{\tilde{e}} - a_{\tilde{e}} \tilde{z}_{\tilde{e}} = \sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \tilde{z}_p - a_{\tilde{e}} \leq \alpha - a_{\tilde{e}}.$$

Next, let $\tilde{z}_{\tilde{e}} = 0$. In this case, we have $\tilde{z}_{\tilde{e}} = 0$. Hence:

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \tilde{z}_p + a_{\tilde{e}} \tilde{z}_{\tilde{e}} - a_{\tilde{e}} \tilde{z}_{\tilde{e}} = \sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \tilde{z}_p \leq \alpha - a_{\tilde{e}} \tilde{z}_{\tilde{e}} \leq \alpha - a_{\tilde{e}}.$$

The last inequality is valid since by assumption $a_{\tilde{e}} < 0$. This completes the proof of validity.

We now show that if $az \leq \alpha$ is facet-defining for $MP_{G'}$, then inequality (35) defines a facet of $MP_G$. Denote by $z^i$, $i = 1, \ldots, k$, the set of all points in $S_{G'}$ satisfying $az = \alpha$. We now convert each of these points to a point $\tilde{z}^i \in MP_G$, by letting $\tilde{z}^i_{\tilde{e}} = 1$ for all $i \in \{1, \ldots, k\}$ and computing $\tilde{z}^i_{\tilde{e}}$ accordingly for every $\tilde{e} \in E(G)$. Clearly, these points satisfy inequality (35) tightly. Since $az \leq \alpha$ defines a facet of $MP_{G'}$, the set $\{\tilde{z}^i : i = 1, \ldots, k\}$ contains $|V(G')| + |E(G')|$ affinely independent points.

By assumption, $\tilde{e} \notin E(G)$, implying that $\dim(MP_G) = \dim(MP_{G'}) + 1$. Thus to complete the proof, we need one point in $S_G$, denoted by $\tilde{z}$, which satisfies (35) tightly and cannot be written as an affine combination of the points $\tilde{z}^i$, $i = 1, \ldots, k$. We now choose a point, say $z^0 \in S_{G'}$, satisfying $az = \alpha$ with $z^0_{\tilde{e}} = 1$. We can always assume that such a point exists, since otherwise the hyperplane $az = \alpha$ is contained in $z_{\tilde{e}} = 0$, which is in contradiction with the assumption that $az \leq \alpha$ defines a facet of $MP_{G'}$ different from $z_{\tilde{e}} \geq 0$. We now lift $z^0$ to a point $\tilde{z} \in S_G$ by
letting \( \hat{z}_0 = 0 \) and \( \hat{z}_e = 0 \). Clearly, this point satisfies (35) tightly and cannot be written as an affine combination of points in \( \hat{z}^i, i = 1, \ldots, k \), since \( z^i_0 = 1 \) for all \( i \). Thus, inequality (35) is facet-defining for MP\(_G\).

In the following example, we demonstrate the applicability of the lifting operation defined in Theorem 7.

Example 12. Consider the hypergraph \( G = (V, E) \) with \( V(G) = \{v_1, v_2, v_3, v_4\} \) and \( E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3, v_4\}\} \). We argue that the following inequality

\[
-z_1 + z_2 + z_3 = 1
\]

defines a facet of MP\(_G\). To see this, consider the graph \( G' \) obtained by removing the node \( v_4 \) from \( G \). It is simple to check that \( -z_1 + z_2 + z_3 = 1 \) defines a facet of MP\(_{G'}\). Since the coefficient of \( z_3 \) in this inequality is negative, by Theorem 7, the inequality (36) is facet-defining for MP\(_G\). For this example, Theorem 5 is not applicable since \( G' \) does not have a node of the form \( \hat{v} \), as defined in this theorem. ◊

We conclude this section by presenting a family of facet-defining inequalities for hypergraphs with a certain structure. The proposed facets are obtained via a recursive application of the lifting operations introduced in this section.

Let \( G = (V, E) \) be a hypergraph with edges \( e_1, \ldots, e_t \), for some \( t \geq 3 \). Suppose that \( e_i, i \in \{1, \ldots, t\} \) has nonempty intersections with \( e_{i-1} \) and \( e_{i+1} \) only, where we define \( e_0 = e_t \) and \( e_{t+1} = e_1 \). In addition, each node is contained in at most two edges of \( G \). Observe that if every edge in \( G \) has cardinality two, then \( G \) is a chordless cycle. Let \( M \) be a subset of \( E \) of odd cardinality. Denote by \( S_1 \subseteq V(G) \) the set of nodes that are not contained in any edge in \( E \setminus M \), and let \( S_2 \subseteq V(G) \) denote a set of nodes that contains exactly one node in \( e_i \cap e_{i+1} \) for every \( i \in \{1, \ldots, t\} \) with \( e_i, e_{i+1} \in E \setminus M \). Then the following inequality is facet-defining for MP\(_G\):
In inequality (38), all coefficients corresponding to the nodes at the intersection of two edges in $M$, and in the intersection of one edge in $M$ and one in $E \setminus M$, are nonnegative. As a result, we can apply Theorem 5 recursively to obtain the following facet-defining inequality for $MP_{G''}$:

$$\sum_{v \in S_1 \cap V(G'')} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq k' + \lfloor |M|/2 \rfloor, \quad (39)$$

where $k' = |S_1 \cap V(G'')| - |\{ i \in \{1, \ldots, t\} : e_i, e_{i+1} \in M \}|$.

Since in inequality (39), the coefficients corresponding to edges in $E \setminus M$ are nonnegative, we can recursively apply Theorem 6 and obtain the following facet-defining inequality for $MP_{G'}$:

$$\sum_{v \in S_1 \cap V(G')} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq k' + \lfloor |M|/2 \rfloor. \quad (40)$$

Finally, observe that in inequality (40), all coefficients corresponding to edges in $M$ are negative. Hence, by a recursive application of Theorem 7, we conclude that inequality (37) defines a facet of $MP_G$.

6 Concluding remarks

We studied the convex hull of a set defined by a collection of multilinear equations from a polyhedral point of view. We developed the theory of various types of lifting operations for this set, giving rise to many types of facet-defining inequalities in the space of the original variables. In particular, together with the known families of facet-defining inequalities for the Boolean quadric polytope, the proposed lifting techniques enable us to construct sharper polyhedral relaxations for mixed-integer nonlinear optimization problems containing multilinear subexpressions. Devising efficient separation algorithms along with extensive computational experimentations with the proposed cutting planes is a subject of future research.

References


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